Traces for Hilbert Complexes

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in Honor of **Patrick Ciarlet's 60th** birthday

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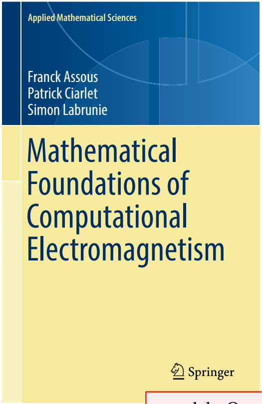
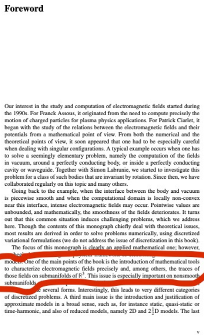
Traces for: Maxwell's Equations / De Rham Complex

⚠ ARE COMPLICATED ⚠





Foreword

Our interest in the study and computation of electromagnetic fields started during the 1990s. For Franck Assous, it originated from the need to compute precisely the motion of charged particles for plasma physics applications. For Patrick Ciarlet, it began with the study of the relations between the electromagnetic fields and their potentials from a mathematical point of view. From both the numerical and the theoretical points of view, it soon appeared that one had to be especially careful when dealing with singular configurations. A typical example occurs when one has to solve a seemingly elementary problem, namely the computation of the fields in vacuum, around a perfectly conducting body, or inside a perfectly conducting cavity or waveguide. Together with Simon Labrunie, we started to investigate this problem for a class of such bodies that are invariant by rotation. Since then, we have collaborated regularly on this topic and many others.

Going back to the example, when the interface between the body and vacuum is piecewise smooth and when the computational domain is locally non-convex near this interface, intense electromagnetic fields may occur. Pointwise values are unbounded, and mathematically, the smoothness of the fields deteriorates. It turns out that this common situation induces challenging problems, which we address here. Though the contents of this monograph chiefly deal with theoretical issues, most results are derived in order to solve problems numerically, using discretized variational formulations (we do not address the issue of discretization in this book).

The focus of this monograph is clearly an applied mathematical one; however, ... models. One of the main points of the book is the introduction of mathematical tools to characterize electromagnetic fields precisely and, among others, the traces of those fields on submanifolds of $\mathbb{R}^3$. This issue is especially important on nonsmooth submanifolds ... several forms. Interestingly, this leads to very different categories of discretized problems. A third main issue is the introduction and justification of approximate models in a broad sense, such as, for instance static, quasi-static or time-harmonic, and also of reduced models, namely 2D and $2\frac{1}{2}$D models. The last

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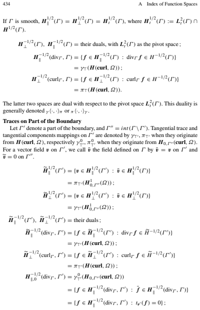


434 A Index of Function Spaces

If Γ is smooth, $\boldsymbol{H}_{\parallel}^{1/2}(\Gamma) = \boldsymbol{H}_{\perp}^{1/2}(\Gamma) = \boldsymbol{H}_{t}^{1/2}(\Gamma)$, where $\boldsymbol{H}_{t}^{1/2}(\Gamma) := \boldsymbol{L}_{t}^{2}(\Gamma) \cap \boldsymbol{H}^{1/2}(\Gamma)$.

$$\begin{aligned}
\boldsymbol{H}_{\perp}^{-1/2}(\Gamma),\ \boldsymbol{H}_{\parallel}^{-1/2}(\Gamma) &= \text{their duals, with } \boldsymbol{L}_t^2(\Gamma) \text{ as the pivot space};\\
\boldsymbol{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) &= \{\boldsymbol{f} \in \boldsymbol{H}_{\parallel}^{-1/2}(\Gamma) \,:\, \operatorname{div}_\Gamma \boldsymbol{f} \in H^{-1/2}(\Gamma)\}\\
&= \gamma_\top(\boldsymbol{H}(\mathbf{curl}, \Omega));\\
\boldsymbol{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma) &= \{\boldsymbol{f} \in \boldsymbol{H}_{\perp}^{-1/2}(\Gamma) \,:\, \operatorname{curl}_\Gamma \boldsymbol{f} \in H^{-1/2}(\Gamma)\}\\
&= \pi_\top(\boldsymbol{H}(\mathbf{curl}, \Omega)).
\end{aligned}$$

The latter two spaces are dual with respect to the pivot space $\boldsymbol{L}_t^2(\Gamma)$. This duality is generally denoted ${}_\gamma\langle\cdot,\cdot\rangle_\pi$ or ${}_\pi\langle\cdot,\cdot\rangle_\gamma$.

Traces on Part of the Boundary

Let Γ' denote a part of the boundary, and $\Gamma'' = int(\Gamma \setminus \Gamma')$. Tangential trace and tangential components mappings on Γ' are denoted by $\gamma_{\top'}, \pi_{\top'}$ when they originate from $\boldsymbol{H}(\mathbf{curl}, \Omega)$, respectively $\gamma^0_{\top'}, \pi^0_{\top'}$ when they originate from $\boldsymbol{H}_{0,\Gamma''}(\mathbf{curl}, \Omega)$. For a vector field $\boldsymbol{v}$ on Γ', we call $\tilde{\boldsymbol{v}}$ the field defined on Γ by $\tilde{\boldsymbol{v}} = \boldsymbol{v}$ on Γ' and $\widetilde{\boldsymbol{v}} = 0$ on Γ''.

$$\begin{aligned}
\widetilde{\boldsymbol{H}}_{\parallel}^{1/2}(\Gamma') &= \{\boldsymbol{v} \in \boldsymbol{H}_{\parallel}^{1/2}(\Gamma') \,:\, \tilde{\boldsymbol{v}} \in \boldsymbol{H}_{\parallel}^{1/2}(\Gamma)\}\\
&= \pi_{\top'}(\boldsymbol{H}^1_{0,\Gamma''}(\Omega));\\
\widetilde{\boldsymbol{H}}_{\perp}^{1/2}(\Gamma') &= \{\boldsymbol{v} \in \boldsymbol{H}_{\perp}^{1/2}(\Gamma') \,:\, \tilde{\boldsymbol{v}} \in \boldsymbol{H}_{\perp}^{1/2}(\Gamma)\}\\
&= \gamma_{\top'}(\boldsymbol{H}^1_{0,\Gamma''}(\Omega));\\
\widetilde{\boldsymbol{H}}_{\parallel}^{-1/2}(\Gamma'),\ \widetilde{\boldsymbol{H}}_{\perp}^{-1/2}(\Gamma') &= \text{their duals};\\
\widetilde{\boldsymbol{H}}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma') &= \{\boldsymbol{f} \in \widetilde{\boldsymbol{H}}_{\parallel}^{-1/2}(\Gamma') \,:\, \operatorname{div}_\Gamma \boldsymbol{f} \in \widetilde{H}^{-1/2}(\Gamma')\}\\
&= \gamma_{\top'}(\boldsymbol{H}(\mathbf{curl}, \Omega));\\
\widetilde{\boldsymbol{H}}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma') &= \{\boldsymbol{f} \in \widetilde{\boldsymbol{H}}_{\perp}^{-1/2}(\Gamma') \,:\, \operatorname{curl}_\Gamma \boldsymbol{f} \in \widetilde{H}^{-1/2}(\Gamma')\}\\
&= \pi_{\top'}(\boldsymbol{H}(\mathbf{curl}, \Omega));\\
\boldsymbol{H}_{\parallel,0}^{-1/2}(\operatorname{div}_\Gamma, \Gamma') &= \gamma^0_{\top'}(\boldsymbol{H}_{0,\Gamma''}(\mathbf{curl}, \Omega))\\
&= \{\boldsymbol{f} \in \boldsymbol{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma') \,:\, \tilde{\boldsymbol{f}} \in \boldsymbol{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)\}\\
&= \{\boldsymbol{f} \in \boldsymbol{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma') \,:\, t_{\boldsymbol{v}'}(\boldsymbol{f}) = 0\};
\end{aligned}$$

lots of pages on traces

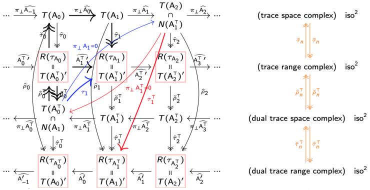
models. One of the main points of the book is the introduction of mathematical tools to characterize electromagnetic fields precisely and, among others, the traces of those fields on submanifolds of $\mathbb{R}^3$. This issue is especially important on nonsmooth submanifolds. Another important issue is the mathematical measure of those fields,

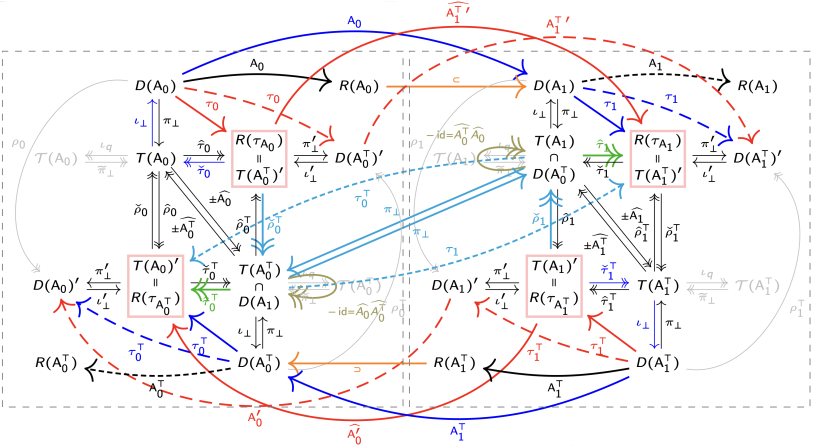
Traces for Hilbert Complexes

Question: Why are traces so complicated?

Question: What is $\widetilde{\mathsf{H}}^{\pm 1/2}_{\parallel/\perp/0(0)}(\operatorname{div}_\Gamma/\operatorname{curl}_\Gamma,\Gamma)$?

... some answers below ...





Traces for Hilbert Complexes

OVERVIEW and BASIC IDEAS

paper in JFA 2023:

R. Hiptmair, D. Pauly, and E. Schulz: *Traces for Hilbert Complexes*

Traces

? Traces ?

Traces without any regularity of the domain?

Is this even possible?

even better:

? Traces ?

Traces without domains (or boundaries)?

Traces

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ ldd c: lin, dendef, cl

Traces for $D(\mathrm{A})$?

$\Omega \subset \mathbb{R}^N$ Lipschitz:

very classical

$D(\mathrm{A}) = \mathrm{H}^1$ or $\mathrm{W}^{1,p}$, scalar trace $u_\mathrm{s} = u|_\Gamma$

classical (we stay in Hilbert spaces)

$D(\mathrm{A}) = \mathrm{H}(\mathrm{curl})$ or $\mathrm{H}(\mathrm{div})$, tan or nor traces $v_\mathrm{t} = (\nu \times v \times \nu)\big|_\Gamma$, $v_\mathrm{n} = (\nu \cdot v)\big|_\Gamma$

more recent (BGG, zoo of complexes)

$$D(\mathrm{A}) = \mathrm{H}(\mathrm{Curl}^\top \mathrm{Curl}_\mathbb{S}), \mathrm{H}(\mathrm{div}\,\mathrm{Div}_\mathbb{S}), \mathrm{H}(\mathrm{sym}\,\mathrm{Curl}_\mathbb{T}), \dots$$
$$\dots \mathrm{H}(\mathrm{Curl}\,\mathrm{Curl}\,\mathrm{Curl}), \mathrm{H}(\mathrm{curl}\,\mathrm{Div}), \mathrm{H}(\mathrm{Grad}\,\mathrm{curl}) \dots$$

$\boxed{\text{traces?}}$

Traces

$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1$ lddc

Traces for $D(\mathrm{A})$?

$\Omega \subset \mathbb{R}^N$ Lipschitz

What if less regularity? What if

- $\boxed{\Omega \text{ just open / no regularity}}$ and $D(\mathrm{A}) = \mathrm{H}^1(\Omega), \mathrm{H}(\mathrm{curl}, \Omega), \mathrm{H}(\mathrm{div}, \Omega), \ldots$?
- $\boxed{\text{no } \Omega}$ at all, just $D(\mathrm{A})$?

Traces

$$\mathrm{A} : D(\mathrm{A}) \subset \mathrm{H}_0 \to \mathrm{H}_1 \qquad \text{lddc}$$

$$\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathrm{H}_1 \to \mathrm{H}_0 \qquad \text{lddc, Hilbert space adjoint}$$

Traces for $D(\mathrm{A})$?

basic idea: integration by parts / extension of adjoints

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \qquad \boxed{\langle y, \mathrm{A}\,x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^*\,y, x\rangle_{\mathrm{H}_0} = 0}$$

think of $\mathrm{A} = \mathring{\mathrm{grad}} : D(\mathrm{A}) = \mathring{\mathrm{H}}^1 \subset \mathrm{L}^2 \to \mathrm{L}^2$

and $\mathrm{A}^* = -\,\mathrm{div} : D(\mathrm{A}^*) = \mathrm{H}(\mathrm{div}) \subset \mathrm{L}^2 \to \mathrm{L}^2$

$$\langle y, \mathring{\mathrm{grad}}\,x\rangle_{\mathrm{L}^2} + \langle \mathrm{div}\,y, x\rangle_{\mathrm{L}^2} = 0$$

Traces

$\mathring{\mathrm{A}} \subset \mathrm{A}$ ldddc

$\mathrm{A}^* \subset \mathrm{A}^\top := \mathring{\mathrm{A}}^*$ ($\mathrm{A}^\top$ formal transpose of A) ldddc, Hilbert space adjoints

Traces for $D(\mathrm{A})$?

basic idea and setting: integration by parts / extension of adjoints

$$\exists\, x \in D(\mathrm{A}) \quad \exists\, y \in D(\mathrm{A}^\top) \qquad \boxed{\langle y, \mathrm{A}\, x\rangle_{\mathrm{H}_1} - \langle \mathrm{A}^\top y, x\rangle_{\mathrm{H}_0} \neq 0}$$

think of $\mathring{\mathrm{grad}} = \mathring{\mathrm{A}} \subset \mathrm{A} = \mathrm{grad}$ $\qquad D(\mathring{\mathrm{grad}}) = \mathring{\mathrm{H}}^1 \subset D(\mathrm{grad}) = \mathrm{H}^1$

and $-\mathring{\mathrm{div}} = \mathrm{grad}^* = \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* = \mathring{\mathrm{grad}}^* = -\,\mathrm{div}$ $\qquad D(\mathring{\mathrm{div}}) = \mathring{\mathrm{H}}(\mathrm{div}) \subset D(\mathrm{div}) = \mathrm{H}(\mathrm{div})$

$$\langle y, \mathrm{grad}\, x\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathrm{div}\, y, x\rangle_{\mathrm{L}^2(\Omega)} = \underbrace{\langle y_{\mathrm{n}}, x_{\mathrm{s}}\rangle_{``\mathrm{L}^2(\Gamma)"}}\ \neq 0$$

$$= ``\int_\Gamma y_{\mathrm{n}} x_{\mathrm{s}}" = \langle\!\langle y_{\mathrm{n}}, x_{\mathrm{s}}\rangle\!\rangle_{\mathrm{H}^{-1/2}(\Gamma),\mathrm{H}^{1/2}(\Gamma)}$$

for some $x \in \mathrm{H}^1$, $y \in \mathrm{H}(\mathrm{div})$

For simplicity of this talk: real Hilbert spaces

Traces

$\mathring{A} \subset A$ lddc

$A^* \subset A^\top = \mathring{A}^*$ lddc, Hilbert space adjoints

$(\mathring{A}, A^*)$ pair "with" boundary conditions (A, A^*), $(\mathring{A}, A^\top = \mathring{A}^*)$

$(A, A^\top)$ pair "without" boundary conditions dual/adjoint pairs

Traces for $D(A)$?

basic idea and setting: integration by parts / extension of adjoints

bd trace

$$\tau_A : D(A) \to D(A^\top)', \qquad \tau_A x(y) := \langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0}$$

$$x \mapsto \tau_A x$$

$$\boxed{x \in D(A),\ y \in D(A^\top)}$$

bd dual trace

$$\tau_{A^\top} : D(A^\top) \to D(A)', \qquad \tau_{A^\top} y(x) := \langle x, A^\top y\rangle_{H_0} - \langle A\,x, y\rangle_{H_1}$$

$$y \mapsto \tau_{A^\top} y$$

note $\boxed{\tau_{A^\top} y(x) = -\tau_A x(y)}$

equivalently bilinear form on $D(A) \times D(A^\top)$ resp. $D(A^\top) \times D(A)$

$$\langle\langle x, y\rangle\rangle := \tau_A x(y) = -\tau_{A^\top} y(x) = \langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0}$$

Hilbert Complexes

Traces for Hilbert Complexes

- We give Traces for Hilbert Complexes.
- On the other hand Hilbert Complexes are necessary for Traces.

Traces for Single Operators

Traces for Single Operators

Traces for Single Operators (and Adjoints)

$\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ lddc (Hilbert space adjoints)

Traces for $D(A)$ and $D(A^\top)$ — traces come always in pairs

$$\boxed{\tau_A x(y) = \langle y, A\,x\rangle_{H_1} - \langle A^\top y, x\rangle_{H_0}}$$

primal / dual traces $\quad \tau_A : D(A) \to D(A^\top)', \quad \tau_{A^\top} : D(A^\top) \to D(A)'$

primal / dual adjoint traces $\quad \tau_A' : D(A^\top)'' \to D(A)', \quad \tau_{A^\top}' : D(A)'' \to D(A^\top)'$

note: Hilbert spaces $H \overset{\text{here}}{=} D(A) \vee D(A^\top)$ are $\boxed{\text{self-dual}}$ (Riesz) and $\boxed{\text{reflexive}}$

$\Rightarrow$ isometric isomorphisms $\quad \rho_H : H \to H' \quad$ and $\quad \iota_d : H \to H''$

Theorem (kernels, boundedness, and adjoints)

- $N(\tau_A) = D(\mathring{A})$ *and* $N(\tau_{A^\top}) = D(A^*)$ *and* $\|\tau_A\|, \|\tau_{A^\top}\| \leq 1$
- $\tau_A' \iota_d = -\tau_{A^\top}$ *and* $\tau_{A^\top}' \iota_d = -\tau_A$

Remark

$\overline{R(\tau_{A^\top})} = \overline{R(\tau_A')} = N(\tau_A)^\circ = D(\mathring{A})^\circ$ and $\overline{R(\tau_A)} = D(A^*)^\circ$

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc (Hilbert space adjoints)

Let $x \in D(\mathrm{A})$. What is / solves

$$\check{y} := -\rho_{D(\mathrm{A}^\top)}^{-1} \tau_{\mathrm{A}} x \in D(\mathrm{A}^\top) \quad \text{and} \quad \check{x} := \mathrm{A}^\top \check{y} \,?$$

Lemma (extension / right inverse)

$$(\check{x}, \check{y}) \in N(\mathrm{A}^\top \mathrm{A} + 1) \times N(\mathrm{A}\,\mathrm{A}^\top + 1) \quad \textit{and} \quad \check{x} - x \in D(\mathring{\mathrm{A}}) = N(\tau_{\mathrm{A}})$$

$$\Rightarrow \quad \boxed{\tau_{\mathrm{A}} \mathrm{A}^\top \check{y} = \tau_{\mathrm{A}} \check{x} = \tau_{\mathrm{A}} x}$$

$$\Rightarrow \quad \boxed{-\tau_{\mathrm{A}} \mathrm{A}^\top \rho_{D(\mathrm{A}^\top)}^{-1} = \mathrm{id}_{R(\tau_{\mathrm{A}})}} \quad \Rightarrow \quad -\mathrm{A}^\top \rho_{D(\mathrm{A}^\top)}^{-1} \ \textit{right inverse of } \tau_{\mathrm{A}} \textit{ on } R(\tau_{\mathrm{A}})$$

note: $$\left(\begin{bmatrix} 0 & -\mathrm{A}^\top \\ \mathrm{A} & 0 \end{bmatrix} + 1 \right) \begin{bmatrix} \check{x} \\ \check{y} \end{bmatrix} = 0$$ (formally skew-symmetric)

Traces for Single Operators (Riesz Isometric Isometries)

$\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ lddc (Hilbert space adjoints)

Lemma (extensions / right inverses)

$-\tau_{\mathrm{A}} \, \mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)} = \mathrm{id}_{R(\tau_{\mathrm{A}})}$ *and* $\check{\tau}_{\mathrm{A}} := -\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)}$ *right inverse of* τ_{A} *on* $R(\tau_{\mathrm{A}})$

$-\tau_{\mathrm{A}^\top} \, \mathrm{A} \, \rho^{-1}_{D(\mathrm{A})} = \mathrm{id}_{R(\tau_{\mathrm{A}^\top})}$ *and* $\check{\tau}_{\mathrm{A}^\top} := -\mathrm{A} \, \rho^{-1}_{D(\mathrm{A})}$ *right inverse of* $\tau_{\mathrm{A}^\top}$ *on* $R(\tau_{\mathrm{A}^\top})$

Definition (extensions / right inverses)

Let $\phi \in R(\tau_{\mathrm{A}})$ and $\psi \in R(\tau_{\mathrm{A}^\top})$. We call:

- $\check{\phi} = -\rho^{-1}_{D(\mathrm{A}^\top)} \phi \in N(\mathrm{A} \, \mathrm{A}^\top + 1)$ harm Neumann ext of ϕ since $\tau_{\mathrm{A}} \, \mathrm{A}^\top \check{\phi} = \phi$
- $\check{\check{\phi}} = \mathrm{A}^\top \check{\phi} = -\mathrm{A}^\top \rho^{-1}_{D(\mathrm{A}^\top)} \phi \in N(\mathrm{A}^\top \mathrm{A} + 1)$ harm Dirichlet ext of ϕ since $\tau_{\mathrm{A}} \check{\check{\phi}} = \phi$
- $\check{\psi} = -\rho^{-1}_{D(\mathrm{A})} \psi \in N(\mathrm{A}^\top \mathrm{A} + 1)$ harm Neumann ext of ψ since $\tau_{\mathrm{A}^\top} \, \mathrm{A} \, \check{\psi} = \psi$
- $\check{\check{\psi}} = \mathrm{A} \, \check{\psi} = -\mathrm{A} \, \rho^{-1}_{D(\mathrm{A})} \psi \in N(\mathrm{A} \, \mathrm{A}^\top + 1)$ harm Dirichlet ext of ψ since $\tau_{\mathrm{A}^\top} \check{\check{\psi}} = \psi$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Theorem (kernels, ranges = annihilators)

- $N(\tau_{\mathrm{A}}) = D(\mathring{\mathrm{A}})$
- $R(\tau_{\mathrm{A}}) = D(\mathrm{A}^*)^\circ = \big\{\Phi \in D(\mathrm{A}^\top)' : D(\mathrm{A}^*) \subset N(\Phi)\big\}$
- $N(\tau_{\mathrm{A}^\top}) = D(\mathrm{A}^*)$
- $R(\tau_{\mathrm{A}^\top}) = D(\mathring{\mathrm{A}})^\circ = \big\{\Phi \in D(\mathrm{A})' : D(\mathring{\mathrm{A}}) \subset N(\Phi)\big\}$

In particular, the kernels and ranges are closed.

Definition and Lemma (trace spaces)

- $T(\mathrm{A}) := D(\mathring{\mathrm{A}})^{\perp_{D(\mathrm{A})}} = N(\mathrm{A}^\top \mathrm{A} + 1) \cong D(\tau_{\mathrm{A}})/N(\tau_{\mathrm{A}}) = D(\mathrm{A})/D(\mathring{\mathrm{A}}) =: \mathcal{T}(\mathrm{A})$
- $T(\mathrm{A}^\top) := D(\mathrm{A}^*)^{\perp_{D(\mathrm{A}^\top)}} = N(\mathrm{A}\,\mathrm{A}^\top + 1) \cong D(\tau_{\mathrm{A}^\top})/N(\tau_{\mathrm{A}^\top}) = D(\mathrm{A}^\top)/D(\mathrm{A}^*) =: \mathcal{T}(\mathrm{A}^\top)$

$\Rightarrow$ red traces $\boxed{\hat{\tau}_{\mathrm{A}} := \tau_{\mathrm{A}}|_{T(\mathrm{A})} : T(\mathrm{A}) \to R(\tau_{\mathrm{A}})}$ $\boxed{\hat{\tau}_{\mathrm{A}^\top} := \tau_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} : T(\mathrm{A}^\top) \to R(\tau_{\mathrm{A}^\top})}$

Theorem (ranges and trace isometries)

$$R(\hat{\tau}_{\mathrm{A}}) = R(\tau_{\mathrm{A}}) = D(\mathrm{A}^*)^\circ = \rho_{D(\mathrm{A}^\top)}\, T(\mathrm{A}^\top) = T(\mathrm{A}^\top)'$$

$$R(\hat{\tau}_{\mathrm{A}^\top}) = R(\tau_{\mathrm{A}^\top}) = D(\mathring{\mathrm{A}})^\circ = \rho_{D(\mathrm{A})}\, T(\mathrm{A}) = T(\mathrm{A})'$$

The reduced traces are isometric isomorphisms.

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Remark (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$
\begin{aligned}
\tau_{\mathrm{A}} &: D(\mathrm{A}) \to R(\tau_{\mathrm{A}}) \subset D(\mathrm{A}^\top)', & \rho_{\mathrm{A}} &:= \rho_{D(\mathrm{A})} : D(\mathrm{A}) \twoheadrightarrow D(\mathrm{A})' \\
\tau_{\mathrm{A}^\top} &: D(\mathrm{A}^\top) \to R(\tau_{\mathrm{A}^\top}) \subset D(\mathrm{A})', & \rho_{\mathrm{A}^\top} &:= \rho_{D(\mathrm{A}^\top)} : D(\mathrm{A}^\top) \twoheadrightarrow D(\mathrm{A}^\top)' \\
\widehat{\tau}_{\mathrm{A}} = \tau_{\mathrm{A}}|_{T(\mathrm{A})} &: T(\mathrm{A}) \twoheadrightarrow R(\tau_{\mathrm{A}}) = T(\mathrm{A}^\top)', & \widehat{\rho}_{\mathrm{A}} &:= \rho_{\mathrm{A}}|_{T(\mathrm{A})} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A})' \\
\widehat{\tau}_{\mathrm{A}^\top} = \tau_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} &: T(\mathrm{A}^\top) \twoheadrightarrow R(\tau_{\mathrm{A}^\top}) = T(\mathrm{A})' & \widehat{\rho}_{\mathrm{A}^\top} &:= \rho_{\mathrm{A}^\top}|_{T(\mathrm{A}^\top)} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A}^\top)'
\end{aligned}
$$

Lemma (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$
\begin{aligned}
&R(\tau_{\mathrm{A}}) = R(\widehat{\tau}_{\mathrm{A}}) = R(\widehat{\rho}_{\mathrm{A}^\top}) = T(\mathrm{A}^\top)', && \widehat{\tau}_{\mathrm{A}} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A}^\top)', && \widehat{\rho}_{\mathrm{A}^\top} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A}^\top)' \\
&R(\tau_{\mathrm{A}^\top}) = R(\widehat{\tau}_{\mathrm{A}^\top}) = R(\widehat{\rho}_{\mathrm{A}}) = T(\mathrm{A})', && \widehat{\tau}_{\mathrm{A}^\top} : T(\mathrm{A}^\top) \twoheadrightarrow T(\mathrm{A})', && \widehat{\rho}_{\mathrm{A}} : T(\mathrm{A}) \twoheadrightarrow T(\mathrm{A})'
\end{aligned}
$$

Definition (inverses of trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$
\begin{aligned}
\check{\tau}_{\mathrm{A}} &:= \widehat{\tau}_{\mathrm{A}}^{-1} : T(\mathrm{A}^\top)' \twoheadrightarrow T(\mathrm{A}), & \check{\rho}_{\mathrm{A}^\top} &:= \widehat{\rho}_{\mathrm{A}^\top}^{-1} : T(\mathrm{A}^\top)' \twoheadrightarrow T(\mathrm{A}^\top) \\
\check{\tau}_{\mathrm{A}^\top} &:= \widehat{\tau}_{\mathrm{A}^\top}^{-1} : T(\mathrm{A})' \twoheadrightarrow T(\mathrm{A}^\top), & \check{\rho}_{\mathrm{A}} &:= \widehat{\rho}_{\mathrm{A}}^{-1} : T(\mathrm{A})' \twoheadrightarrow T(\mathrm{A})
\end{aligned}
$$

Remark

Continuity of traces and extensions for free! (no ass on $R(\mathrm{A})$ or domains Ω)

⚠ EVEN ISOMETRIES ⚠

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

Remark (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

Continuity/isometry of traces and extensions for free! (no ass on $R(\mathrm{A})$ or domains Ω)

⚠ $\boxed{\text{NO compact embeddings or Friedrichs/Poincaré type estimates NEEDED}}$ ⚠

$\boxed{\text{We do NOT need:}}$

e.g.: $\Omega \subset \mathbb{R}^3$ bd, weak Lip (compact embeddings)

(Weck 1972/'74, Weber 1980, Picard 1984)

$$\mathrm{H}^1(\Omega) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Omega) \quad \text{or} \quad \mathring{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathrm{H}(\mathrm{div}, \Omega) \hookrightarrow\!\!\!\!\rightarrow \mathrm{L}^2(\Omega)$$

or even weaker (Friedrichs/Poincaré type estimate $\Leftrightarrow$ closed range)

Ω (weak Lip, bd in ONE direction) $\Rightarrow$ $R(\mathring{\mathrm{grad}})$ and $R(\mathrm{div})$ closed

Ω (weak Lip, bd in TWO directions) $\Rightarrow$ $R(\mathring{\mathrm{curl}})$ and $R(\mathrm{curl})$ closed

Ω (weak Lip, bd in THREE directions) $\Rightarrow$ $R(\mathring{\mathrm{div}})$ and $R(\mathrm{grad})$ closed

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

Theorem (trace /Riesz isometric isomorphisms $\twoheadrightarrow$)

$$T(\mathrm{A})' \cong_{\widehat{\rho}_\mathrm{A}} T(\mathrm{A}) \cong_{\widehat{\tau}_\mathrm{A}} T(\mathrm{A}^\top)', \qquad T(\mathrm{A}^\top)' \cong_{\widehat{\rho}_{\mathrm{A}^\top}} T(\mathrm{A}^\top) \cong_{\widehat{\tau}_{\mathrm{A}^\top}} T(\mathrm{A})'$$

bilinear (sesquilinear) forms on $T(\mathrm{A}) \times T(\mathrm{A}^\top)$ or $D(\mathrm{A}) \times D(\mathrm{A}^\top)$

$$\langle\!\langle x, y \rangle\!\rangle := \langle\!\langle x, y \rangle\!\rangle_\tau := \tau_\mathrm{A} x(y) = -\tau_{\mathrm{A}^\top} y(x) = \langle \mathrm{A}\, x, y \rangle_{\mathrm{H}_1} - \langle x, \mathrm{A}^\top y \rangle_{\mathrm{H}_0},$$
$$\langle\!\langle x, y \rangle\!\rangle_\rho := \rho_\mathrm{A} x(y) = \langle x, y \rangle_{D(\mathrm{A})} = \langle x, y \rangle_{\mathrm{H}_0} + \langle \mathrm{A}\, x, \mathrm{A}\, y \rangle_{\mathrm{H}_1}$$

Corollary (“integration by parts”)

$$\langle \mathrm{A}\, x, y \rangle_{\mathrm{H}_1} = \langle x, \mathrm{A}^\top y \rangle_{\mathrm{H}_0} + \langle\!\langle x, y \rangle\!\rangle$$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (Iddc)

Isometric Isomorphisms ($\twoheadrightarrow$)

$$
\begin{tikzcd}
 & D(\mathrm{A}) \arrow[d, "\pi_\perp"'] & & \\
\mathcal{T}(\mathrm{A}) & T(\mathrm{A}) \arrow[l, twoheadrightarrow, "\iota_q"'] \arrow[r, twoheadrightarrow, "\widehat{\tau}_\mathrm{A}"] \arrow[d, twoheadrightarrow, "\widehat{\rho}_\mathrm{A}"'] & R(\tau_\mathrm{A}) = T(\mathrm{A}^\top)' & \\
 & T(\mathrm{A})' = R(\tau_{\mathrm{A}^\top}) & T(\mathrm{A}^\top) \arrow[l, twoheadrightarrow, "\widehat{\tau}_{\mathrm{A}^\top}"] \arrow[u, twoheadrightarrow, "\widehat{\rho}_{\mathrm{A}^\top}"'] \arrow[r, twoheadrightarrow, "\iota_q"'] & \mathcal{T}(\mathrm{A}^\top) \\
 & & D(\mathrm{A}^\top) \arrow[u, "\pi_\perp"'] &
\end{tikzcd}
$$

$D(\mathrm{A}) = D(\mathring{\mathrm{A}}) \oplus_{D(\mathrm{A})} T(\mathrm{A})$

$[x_\perp] = [x]$ and $\tau_\mathrm{A} x_\perp = \tau_\mathrm{A} x = \tau_\mathrm{A}[x]$

Traces for Single Operators $\quad \mathring{\mathrm{A}} \subset \mathrm{A} \quad$ and $\quad \mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^* \quad$ (lddc)

Isometric Isomorphisms ($\twoheadrightarrow$)

$$
\begin{tikzcd}
 & D(\mathrm{A}) \arrow[rr, "\mathrm{A}"] \arrow[d, "\pi_\perp"'] \arrow[dr, "\tau_{\mathrm{A}}"] & & R(\mathrm{A}) \\
\mathcal{T}(\mathrm{A}) & T(\mathrm{A}) \arrow[l, two heads, "\iota_q"'] \arrow[r, two heads, "\hat{\tau}_{\mathrm{A}}"] \arrow[d, two heads, "\hat{\rho}_{\mathrm{A}}"'] \arrow[dr, two heads, "\hat{\mathrm{A}}"] & R(\tau_{\mathrm{A}}) = T(\mathrm{A}^\top)' & \\
 & T(\mathrm{A})' = R(\tau_{\mathrm{A}^\top}) & T(\mathrm{A}^\top) \arrow[l, two heads, "\hat{\tau}_{\mathrm{A}^\top}"] \arrow[u, two heads, "\hat{\rho}_{\mathrm{A}^\top}"'] \arrow[r, two heads, "\iota_q"'] & \mathcal{T}(\mathrm{A}^\top) \\
R(\mathrm{A}^\top) & & D(\mathrm{A}^\top) \arrow[ll, "\mathrm{A}^\top"] \arrow[u, "\pi_\perp"'] \arrow[ul, "\tau_{\mathrm{A}^\top}"] &
\end{tikzcd}
$$

$\hat{\mathrm{A}} := \mathrm{A}|_{T(\mathrm{A})}$

Traces for Single Operators $\mathring{\mathrm{A}} \subset \mathrm{A}$ and $\mathrm{A}^* \subset \mathrm{A}^\top = \mathring{\mathrm{A}}^*$ (lddc)

Isometric Isomorphisms ($\twoheadrightarrow$)

$$
\begin{tikzcd}[column sep=large, row sep=large]
 & D(\mathrm{A}) \arrow[rr, bend left, "\mathrm{A}"] \arrow[drr, bend left, "\tau_{\mathrm{A}}"] \arrow[dr, "\tau_{\mathrm{A}}"] \arrow[d, shift left, two heads, "\pi_\perp = \iota_\perp^*"] \arrow[ddl, bend right=60, two heads, "\rho_{\mathrm{A}}"'] & & R(\mathrm{A}) \\
\mathcal{T}(\mathrm{A}) \arrow[r, shift right, two heads, "\widetilde{\pi}_\perp"'] & T(\mathrm{A}) \arrow[l, shift right, two heads, "\iota_q"'] \arrow[u, shift left, "\iota_\perp"] \arrow[r, shift left, two heads, "\widehat{\tau}_{\mathrm{A}}"] \arrow[d, shift left, two heads, "\widehat{\rho}_{\mathrm{A}}"] \arrow[dr, shift left, two heads, "\pm\widehat{\mathrm{A}}"] & R(\tau_{\mathrm{A}}) = T(\mathrm{A}^\top)' \arrow[l, shift left, two heads, "\check{\tau}_{\mathrm{A}}"] \arrow[r, shift left, "\pi_\perp'"] \arrow[d, shift left, two heads, "\check{\rho}_{\mathrm{A}^\top}"] & D(\mathrm{A}^\top)' \arrow[l, shift left, "\iota_\perp'"] \\
D(\mathrm{A})' \arrow[r, shift right, "\iota_\perp'"'] & T(\mathrm{A})' = R(\tau_{\mathrm{A}^\top}) \arrow[l, shift right, two heads, "\pi_\perp'"'] \arrow[u, shift left, two heads, "\check{\rho}_{\mathrm{A}}"] \arrow[r, shift left, two heads, "\check{\tau}_{\mathrm{A}^\top}"] & T(\mathrm{A}^\top) \arrow[l, shift left, two heads, "\widehat{\tau}_{\mathrm{A}^\top}"] \arrow[ul, shift left, two heads, "\pm\widehat{\mathrm{A}^\top}"] \arrow[u, shift left, two heads, "\widehat{\rho}_{\mathrm{A}^\top}"] \arrow[r, shift left, two heads, "\iota_q"] \arrow[d, shift left, "\iota_\perp"] & \mathcal{T}(\mathrm{A}^\top) \arrow[l, shift left, two heads, "\widetilde{\pi}_\perp"] \\
R(\mathrm{A}^\top) & & D(\mathrm{A}^\top) \arrow[ll, bend left, "\mathrm{A}^\top"] \arrow[ull, bend right, "\tau_{\mathrm{A}^\top}"'] \arrow[ul, "\tau_{\mathrm{A}^\top}"] \arrow[u, shift left, two heads, "\pi_\perp = \iota_\perp^*"] \arrow[uur, bend right=60, two heads, "\rho_{\mathrm{A}^\top}"'] &
\end{tikzcd}
$$

"on $T(\mathrm{A}) = N(\mathrm{A}^\top \mathrm{A} + 1)$ and $T(\mathrm{A}^\top) = N(\mathrm{A}\,\mathrm{A}^\top + 1)$":

$$
\begin{array}{llll}
\check{\tau}_{\mathrm{A}} = -\widehat{\mathrm{A}}^\top \check{\rho}_{\mathrm{A}^\top} & \widehat{\tau}_{\mathrm{A}} = \widehat{\rho}_{\mathrm{A}^\top} \widehat{\mathrm{A}} & \widehat{\mathrm{A}}^{-1} = -\widehat{\mathrm{A}}^\top & \widehat{\mathrm{A}} = \mathrm{A}|_{T(\mathrm{A})} \\
\check{\tau}_{\mathrm{A}^\top} = -\widehat{\mathrm{A}} \check{\rho}_{\mathrm{A}} & \widehat{\tau}_{\mathrm{A}^\top} = \widehat{\rho}_{\mathrm{A}} \widehat{\mathrm{A}}^\top & (\widehat{\mathrm{A}}^\top)^{-1} = -\widehat{\mathrm{A}} & \widehat{\mathrm{A}}^\top = \mathrm{A}^\top|_{T(\mathrm{A}^\top)}
\end{array}
$$

Traces for Single Operators $\mathring{A} \subset A$ and $A^* \subset A^\top = \mathring{A}^*$ (lddc)

Theorem (kernels and ranges of traces / isometric isomorphisms)

- $N(\tau_A) = D(\mathring{A})$
- $R(\tau_A) = R(\hat{\tau}_A) = D(A^*)^\circ = R(\hat{\rho}_{A^\top}) = T(A^\top)'$
- $T(A) = D(\mathring{A})^{\perp_{D(A)}} = N(A^\top A + 1)$
- $T(A) \cong \mathcal{T}(A) = D(A)/D(\mathring{A})$
- $N(\tau_{A^\top}) = D(A^*)$
- $R(\tau_{A^\top}) = R(\hat{\tau}_{A^\top}) = D(\mathring{A})^\circ = R(\hat{\rho}_A) = T(A)'$
- $T(A^\top) = D(A^*)^{\perp_{D(A^\top)}} = N(A A^\top + 1)$
- $T(A^\top) \cong \mathcal{T}(A^\top) = D(A^\top)/D(A^*)$

Remark (summary)

- trace ranges are annihilators of trace kernels
- trace ranges are duals of reduced transpose trace spaces
- trace spaces are kernels $N(A^\top A + 1)$ and $N(A A^\top + 1)$ ("harmonic fields")
- trace spaces are orthogonal complements of trace kernels (—)
- trace spaces are minimal norm extensions (—)
- trace spaces are quotient spaces of trace kernels

note:

- elements of the trace spaces are "smooth"
- regularity is never a problem (regularity not a good term)
- integrability is always the problem

Traces for Hilbert Complexes

so far NO (Hilbert) complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Traces for Hilbert Complexes

Different Hilbert Complexes

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^\top = \mathring{\mathrm{A}}_0^*}{\overset{\mathring{\mathrm{A}}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^\top = \mathring{\mathrm{A}}_1^*}{\overset{\mathring{\mathrm{A}}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \qquad \text{(unbd prim/dual HilComs)}$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathrm{H}_0 \underset{\mathrm{A}_0^*}{\overset{\mathrm{A}_0}{\rightleftarrows}} \mathrm{H}_1 \underset{\mathrm{A}_1^*}{\overset{\mathrm{A}_1}{\rightleftarrows}} \mathrm{H}_2 \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots \qquad \text{(unbd prim/dual HilComs)}$$

$$\cdots \xrightarrow{\cdots} D(\mathrm{A}_0) \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_1) \xrightarrow{\mathrm{A}_1} D(\mathrm{A}_2) \xrightarrow{\cdots} \cdots \qquad \text{(bd DomCom)}$$

$$\cdots \xleftarrow[\cdots]{} D(\mathrm{A}_0)' \xleftarrow[\mathrm{A}_0']{} D(\mathrm{A}_1)' \xleftarrow[\mathrm{A}_1']{} D(\mathrm{A}_2)' \xleftarrow[\cdots]{} \cdots \qquad \text{(bd adjoint DomCom)}$$

$$\cdots \xleftarrow{\cdots} D(\mathrm{A}_{-1}^\top) \xleftarrow{\mathrm{A}_0^\top} D(\mathrm{A}_0^\top) \xleftarrow{\mathrm{A}_1^\top} D(\mathrm{A}_1^\top) \xleftarrow{\cdots} \cdots \qquad \text{(bd DomCom)}$$

$$\cdots \xrightarrow[\cdots]{} D(\mathrm{A}_{-1}^\top)' \xrightarrow[\mathrm{A}_0^\top{}']{} D(\mathrm{A}_0^\top)' \xrightarrow[\mathrm{A}_1^\top{}']{} D(\mathrm{A}_1^\top)' \xrightarrow[\cdots]{} \cdots \qquad \text{(bd adjoint DomCom)}$$

setting

- $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $\mathrm{A}_\ell^* \subset \mathrm{A}_\ell^\top = \mathring{\mathrm{A}}_\ell^*$ (lddc)
- $R(\mathring{\mathrm{A}}_0) \subset N(\mathring{\mathrm{A}}_1)$, $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$ (prim HilComs)
- $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$, $R(\mathrm{A}_1^\top) \subset N(\mathrm{A}_0^\top)$ (dual HilComs)
- $R(\mathrm{A}_1') \subset N(\mathrm{A}_0')$, $R(\mathrm{A}_0^\top{}') \subset N(\mathrm{A}_1^\top{}')$ (adjoint HilComs)

note $\mathrm{A}_n^\top{}' : D(\mathrm{A}_{n-1}^\top)' \to D(\mathrm{A}_n^\top)'$

$\Rightarrow$ $\mathrm{A}_n^\top{}' \mathrel{\widehat{\ncong}} \mathrm{A}_n : D(\mathrm{A}_n) \to D(\mathrm{A}_{n+1})$ but $\mathrm{A}_n^\top{}' \mathrel{\widehat{\cong}} \mathrm{A}_{n-1} : D(\mathrm{A}_{n-1}) \to D(\mathrm{A}_n)$ (index shift)

Traces for Hilbert Complexes $\mathring{A}_\ell \subset A_\ell$ and $R(\dots) \subset N(\dots)$

$$\boxed{\begin{aligned} A_0 &: D(A_0) \to D(A_1) \\ \tau_{A_0} &: D(A_0) \to D(A_0^\top)' \\ A_1^{\top\prime} &: D(A_0^\top)' \to D(A_1^\top)' \end{aligned}} \qquad \boxed{\begin{aligned} A_1^\top &: D(A_1^\top) \to D(A_0^\top) \\ \tau_{A_1^\top} &: D(A_1^\top) \to D(A_1)' \\ A_0' &: D(A_1)' \to D(A_0)' \end{aligned}} \qquad \begin{aligned} &\text{(vol diff ops)} \\ &\text{(trace ops)} \\ &\text{(surf diff ops)} \end{aligned}$$

Theorem (surface differential operators / commutators with traces)

$$\tau_{A_1} A_0 = -A_1^{\top\prime} \tau_{A_0} \qquad \textit{and} \qquad \tau_{A_0^\top} A_1^\top = -A_0' \tau_{A_1^\top}$$

Theorem (integration by parts . . .)

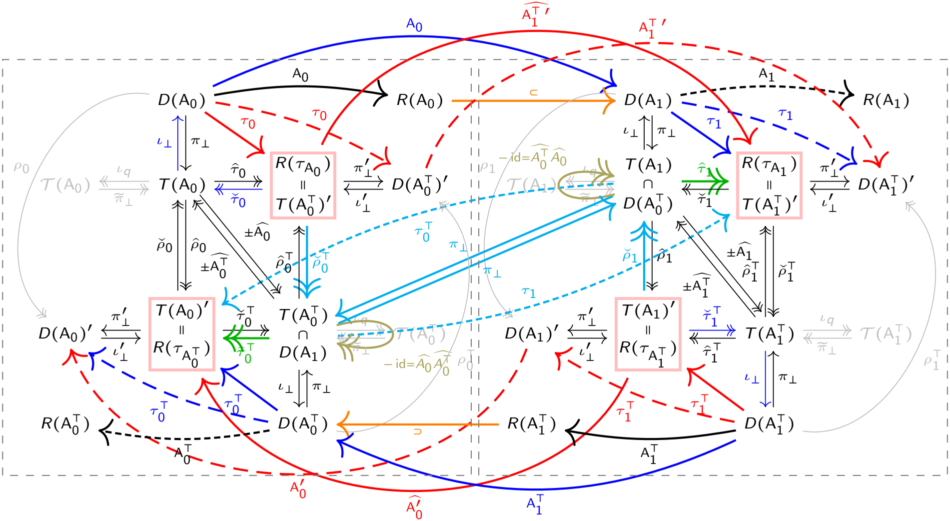
- *. . . on domains:* $x \in D(A)$, $y \in D(A^\top)$ *or* $x \in T(A)$, $y \in T(A^\top)$ $\Rightarrow$ $\langle A x, y\rangle_{H_1} = \langle x, A^\top y\rangle_{H_0} + \langle\!\langle x, y\rangle\!\rangle$
- *. . . on trace domains* $\boxed{\tau_{A_1} A_0 = -A_1^{\top\prime} \tau_{A_0}}$ $x \in D(A_0)$, $z \in D(A_1^\top)$ $\Rightarrow$
$$\langle\!\langle A_0 x, z\rangle\!\rangle_1 = \tau_{A_1}(A_0 x)(z) = -\tau_{A_0}(x)(A_1^\top z) = -\langle\!\langle x, A_1^\top z\rangle\!\rangle_0$$
- *. . . on trace spaces* $\boxed{\hat\tau_{A_1} \pi_\perp \widehat{A_0} = -\widehat{A_1^{\top\prime}} \pi_\perp' \hat\tau_{A_0}}$ $x \in T(A_0)$, $z \in T(A_1^\top)$ $\Rightarrow$
$$\langle\!\langle \pi_\perp \widehat{A_0} x, z\rangle\!\rangle_1 = \hat\tau_{A_1}(\pi_\perp \widehat{A_0} x)(z) = -\hat\tau_{A_0}(x)(\pi_\perp \widehat{A_1^\top} z) = -\langle\!\langle x, \pi_\perp \widehat{A_1^\top} z\rangle\!\rangle_0$$
- *. . . on trace ranges* $\boxed{\widehat{A_1^{\top\prime}} = -\hat\tau_{A_1} \iota_d^{-1} (\widehat{A_0'})' \iota_d \check\tau_{A_0}}$ $\varphi \in R(\tau_{A_0})$, $\psi \in R(\tau_{A_1^\top})$ $\Rightarrow$
$$\langle\!\langle\!\langle \widehat{A_1^{\top\prime}} \varphi, \psi\rangle\!\rangle\!\rangle_1 = \langle\!\langle \check\tau_{A_1} \widehat{A_1^{\top\prime}} \varphi, \check\tau_{A_1^\top} \psi\rangle\!\rangle_1 = -\langle\!\langle \check\tau_{A_0} \varphi, \check\tau_{A_0^\top} \widehat{A_0'} \psi\rangle\!\rangle_0 = -\langle\!\langle\!\langle \varphi, \widehat{A_0'} \psi\rangle\!\rangle\!\rangle_0$$

Traces for Hilbert Complexes $\quad \mathring{A}_\ell \subset A_\ell \quad$ and $\quad R(\dots) \subset N(\dots)$

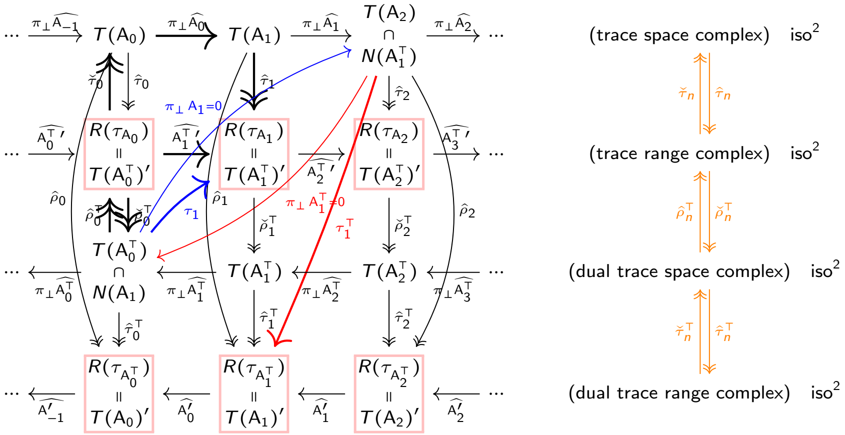
$A_0 : D(A_0) \to D(A_1)$	(vol diff ops)	$A_1^\top : D(A_1^\top) \to D(A_0^\top)$
$A_1^{\top\,\prime} : D(A_0^\top)' \to D(A_1^\top)'$	(surf diff ops)	$A_0' : D(A_1)' \to D(A_0)'$

⚠ CRAZY ⚠

simple idea $\Rightarrow$ amazing complexity



Traces for Hilbert Complexes $\mathring{\mathrm{A}}_\ell \subset \mathrm{A}_\ell$ and $R(\dots) \subset N(\dots)$



Fin . . .

Dear Patrick . . .

Bon Anniversaire!

. . . to be continued . . .

Appendix

Regular Subspaces and Duals

"Regular Subspaces" and Duals

Regular Subspaces and Duals

$$\boxed{\mathrm{A}_0' : D(\mathrm{A}_1)' \to D(\mathrm{A}_0)'}$$

- $\mathrm{H}_1^+ \subset D(\mathrm{A}_1) \subset \mathrm{H}_1$ (bd dense embs of reg subsps)
- $D(\mathrm{A}_1) = \mathrm{H}_1^+ + \mathrm{A}_0\,\mathrm{H}_0^+$ (bd reg deco ops)
- $\mathrm{H}_0^+(\mathrm{A}_0) = \{x \in \mathrm{H}_0^+ : \mathrm{A}_0\,x \in \mathrm{H}_1^+\} \subset \mathrm{H}_0^+ \subset D(\mathrm{A}_0) \subset \mathrm{H}_0$ (bd dense embs)
- $\mathring{\mathrm{H}}_0^- = \mathrm{H}_0^{+\,\prime}$ (duals)
- $\mathrm{H}_1^+ \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^\top) \subset \mathrm{H}_1$ (bd dense embs of reg subsps)

note: $\mathrm{H}_0^+(\mathrm{A}_0) \subset \mathrm{H}_0^+ \subset D(\mathrm{A}_0) \subset \mathrm{H}_0$ and $\mathrm{H}_0' \subset D(\mathrm{A}_0)' \subset \mathring{\mathrm{H}}_0^- \subset \mathrm{H}_0^+(\mathrm{A}_0)'$

$\Rightarrow$ extend A_0' to $\boxed{\mathrm{A}_0' : \mathring{\mathrm{H}}_1^- \to \mathrm{H}_0^+(\mathrm{A}_0)'}$ by

$\forall\, x \in \mathrm{H}_0^+(\mathrm{A}_0) \quad \mathrm{A}_0'\,\psi(x) := \psi(\mathrm{A}_0\,x)$

- $\mathrm{H}_1' \subset D(\mathrm{A}_1)' \overset{!}{=} \boxed{\mathring{\mathrm{H}}_1^-(\mathrm{A}_0') := \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\,\psi \in \mathring{\mathrm{H}}_0^-\}} \subset \mathring{\mathrm{H}}_1^- = \mathrm{H}_1^{+\,\prime} \subset \mathrm{H}_1^+(\mathrm{A}_1)'$

$\subset$: $\mathrm{H}_1^+ \subset D(\mathrm{A}_1) \quad \Rightarrow \quad \psi \in D(\mathrm{A}_1)' \subset \mathring{\mathrm{H}}_1^- \quad$ and $\quad \mathrm{A}_0'\,\psi \in D(\mathrm{A}_0)' \subset \mathring{\mathrm{H}}_0^-$

$\supset$: $\psi \in \mathring{\mathrm{H}}_1^-(\mathrm{A}_0')$ and $D(\mathrm{A}_1) \ni y = y_1 + \mathrm{A}_0\,y_0 \in \mathrm{H}_1^+ + \mathrm{A}_0\,\mathrm{H}_0^+$ (reg deco)

$\Rightarrow \quad \psi y := \psi y_1 + (\mathrm{A}_0'\,\psi)y_0 \quad$ and $\quad |\psi y| \le c|\psi|_{\mathring{\mathrm{H}}_1^-(\mathrm{A}_0')}|y|_{D(\mathrm{A}_1)} \quad \Rightarrow \quad \psi \in D(\mathrm{A}_1)'$

Characterisation of Dual Spaces by Regular Subspaces

Characterisation of Dual Spaces by “Regular Subspaces”

Characterisation of Dual Spaces by Regular Subspaces

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(\mathrm{A}_1)' = \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\psi \in \mathring{\mathrm{H}}_0^-\}$$
$$D(\mathrm{A}_0^\top)' = \mathring{\mathrm{H}}_1^-(\mathrm{A}_1^{\top\prime}) = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_1^{\top\prime}\psi \in \mathring{\mathrm{H}}_2^-\}$$

with equivalent norms.

Theorem (Characterisation of Dual Spaces by Regular Subspaces)

$$D(\mathring{\mathrm{A}}_1)' = \mathrm{H}_1^-(\mathring{\mathrm{A}}_0') = \{\psi \in \mathrm{H}_1^- : \mathring{\mathrm{A}}_0'\psi \in \mathrm{H}_0^-\}$$
$$D(\mathrm{A}_0^*)' = \mathrm{H}_1^-(\mathrm{A}_1^{*\prime}) = \{\psi \in \mathrm{H}_1^- : \mathrm{A}_1^{*\prime}\psi \in \mathrm{H}_2^-\}$$

with equivalent norms.

Characterisation of Trace Ranges by Regular Subspaces

Characterisation of Trace Ranges by “Regular Subspaces”

Characterisation of Trace Ranges by Regular Subspaces

recall traces: $\tau_{\mathrm{A}_0} : D(\mathrm{A}_0) \to D(\mathrm{A}_0^\top)'$, $\quad \tau_{\mathrm{A}_1^\top} : D(\mathrm{A}_1^\top) \to D(\mathrm{A}_1)'$

- $N(\tau_{\mathrm{A}_1^\top}) = D(\mathrm{A}_1^*)$
- $R(\tau_{\mathrm{A}_1^\top}) = D(\mathring{\mathrm{A}}_1)^\circ = \{\psi \in D(\mathrm{A}_1)' : \psi|_{D(\mathring{\mathrm{A}}_1)} = 0\}$
- $N(\tau_{\mathrm{A}_0}) = D(\mathring{\mathrm{A}}_0)$
- $R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^*)^\circ = \{\psi \in D(\mathrm{A}_0^\top)' : \psi|_{D(\mathrm{A}_0^*)} = 0\}$

density of $\quad \mathring{\mathrm{H}}_1^+ \subset D(\mathring{\mathrm{A}}_1)$ and $\overset{*}{\mathrm{H}}{}_1^+ \subset D(\mathrm{A}_0^*)$ $\quad\Rightarrow$

- $R(\tau_{\mathrm{A}_1^\top}) = \mathring{\mathrm{H}}_1^{+\circ}$ as closed subspace of $D(\mathrm{A}_1)'$
- $R(\tau_{\mathrm{A}_0}) = \overset{*}{\mathrm{H}}{}_1^{+\circ}$ as closed subspace of $D(\mathrm{A}_0^\top)'$

$\Rightarrow$ more detailed

Theorem (Characterisation of Trace Ranges by Regular Subspaces)

$$R(\tau_{\mathrm{A}_1^\top}) = D(\mathrm{A}_1)' \cap D(\mathring{\mathrm{A}}_1)^\circ = \mathring{\mathrm{H}}_1^-(\mathrm{A}_0') \cap \mathring{\mathrm{H}}_1^{+\circ} = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_0'\,\psi \in \mathring{\mathrm{H}}_0^- \ \wedge \ \psi|_{\mathring{\mathrm{H}}_1^+} = 0\}$$

$$R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^\top)' \cap D(\mathrm{A}_0^*)^\circ = \mathring{\mathrm{H}}_1^-(\mathrm{A}_1^{\top\prime}) \cap \overset{*}{\mathrm{H}}{}_1^{+\circ} = \{\psi \in \mathring{\mathrm{H}}_1^- : \mathrm{A}_1^{\top\prime}\psi \in \mathring{\mathrm{H}}_2^- \ \wedge \ \psi|_{\overset{*}{\mathrm{H}}{}_1^+} = 0\}$$

with equivalent norms.

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces

Trace Hilbert Complexes

Hilbert Complexes of Traces and Trace Spaces ... to be continued ...

- different unbounded versions of "surface differential operators"

$$\cdots \xrightarrow{\cdots} D(\mathrm{A}_0^\top)' \xrightarrow{\mathrm{A}_1^{\top\,\prime}} D(\mathrm{A}_1^\top)' \xrightarrow{\mathrm{A}_2^{\top\,\prime}} D(\mathrm{A}_2^\top)' \xrightarrow{\cdots} \cdots$$

$$\cdots \xleftarrow[\cdots]{} D(\mathrm{A}_1)' \xleftarrow[\mathrm{A}_1']{} D(\mathrm{A}_2)' \xleftarrow[\mathrm{A}_2']{} D(\mathrm{A}_3)' \xleftarrow[\cdots]{} \cdots$$

$$\cdots \xrightarrow{\cdots} \overset{*}{\mathrm{H}}{}_1^{+\circ} \xrightarrow{\mathrm{A}_1^{\top\,\prime}} \overset{*}{\mathrm{H}}{}_2^{+\circ} \xrightarrow{\mathrm{A}_2^{\top\,\prime}} \overset{*}{\mathrm{H}}{}_3^{+\circ} \xrightarrow{\cdots} \cdots$$

$$\cdots \xleftarrow[\cdots]{} \mathring{\mathrm{H}}{}_1^{+\circ} \xleftarrow[\mathrm{A}_1']{} \mathring{\mathrm{H}}{}_2^{+\circ} \xleftarrow[\mathrm{A}_2']{} \mathring{\mathrm{H}}{}_3^{+\circ} \xleftarrow[\cdots]{} \cdots$$

$$\cdots \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \mathring{\mathrm{H}}{}_1^{-} \underset{\mathrm{A}_1'}{\overset{\mathrm{A}_1^{\top\,\prime}}{\rightleftarrows}} \mathring{\mathrm{H}}{}_2^{-} \underset{\mathrm{A}_2'}{\overset{\mathrm{A}_2^{\top\,\prime}}{\rightleftarrows}} \mathring{\mathrm{H}}{}_3^{-} \underset{\cdots}{\overset{\cdots}{\rightleftarrows}} \cdots$$

three interpretations

- $D(\mathrm{A}_{n+1}^{\top\,\prime}) = R(\tau_{\mathrm{A}_n}) \subset D(\mathrm{A}_n^\top)'$
- $D(\mathrm{A}_n') = R(\tau_{\mathrm{A}_{n+1}^\top}) \subset D(\mathrm{A}_{n+1})'$
- $R(\tau_{\mathrm{A}_0}) = \overset{*}{\mathrm{H}}{}_1^{+\circ}$ cl sbsp of both $D(\mathrm{A}_0^\top)' \subset \mathring{\mathrm{H}}{}_1^-$
- $\mathrm{A}_1^{\top\,\prime} : R(\tau_{\mathrm{A}_0}) \to R(\tau_{\mathrm{A}_1})$
- $R(\tau_{\mathrm{A}_n}) \subset \overset{*}{\mathrm{H}}{}_{n+1}^{+\circ} \subset \mathring{\mathrm{H}}{}_{n+1}^-$
- $R(\tau_{\mathrm{A}_0}) = D(\mathrm{A}_0^\top)' \cap D(\overset{*}{\mathrm{A}}_0)^\circ$

$$= \mathring{\mathrm{H}}{}_1^-(\mathrm{A}_1^{\top\,\prime}) \cap \overset{*}{\mathrm{H}}{}_1^{+\circ}$$

$$= \{\psi \in \mathring{\mathrm{H}}{}_1^- : \mathrm{A}_1^{\top\,\prime}\psi \in \mathring{\mathrm{H}}{}_2^- \wedge \psi|_{\overset{*}{\mathrm{H}}{}_1^+} = 0\}$$

$$= \underbrace{\{\psi \in \overset{*}{\mathrm{H}}{}_1^{+\circ} : \mathrm{A}_1^{\top\,\prime}\psi \in \overset{*}{\mathrm{H}}{}_2^{+\circ}\}}_{=:\overset{*}{\mathrm{H}}{}_1^{+\circ}(\mathrm{A}_1^{\top\,\prime})}$$

- compact embeddings for trace Hilbert complexes
- boundary value problems on trace Hilbert complexes