Theoretical and numerical aspects for the reconstruction of the electrical properties in the human brain by magnetic resonance imaging

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Conference in honor of the 60th birthday of Patrick Ciarlet

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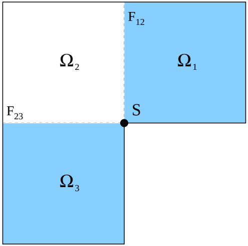
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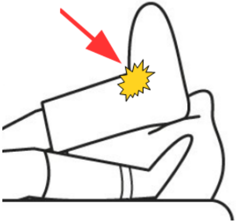
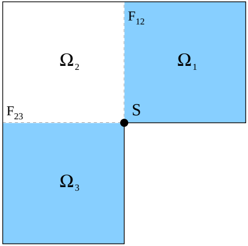
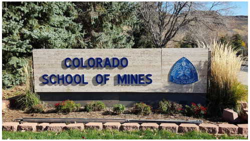
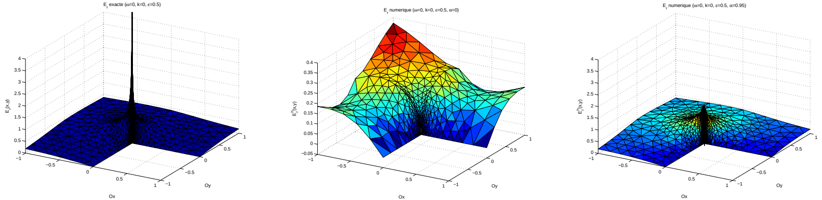


Historical background (1998 – 2010)

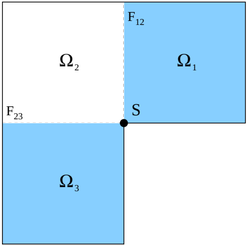


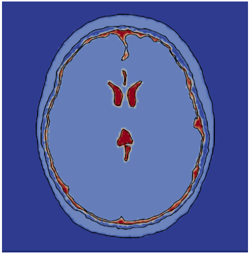




Exact (singular) solution (left). Classical approximation with P1 Finite Elements (middle). Weighted approximation (right).

Transition . . .





Reconstruction of the electrical properties in the human brain by MRI

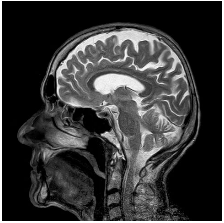
- Medical context
- Formulation of the inverse problem

- Numerical resolution by Contrast Source Inversion

Medical context

Aim

Reconstruct the electrical properties (permittivity ε and conductivity σ) of the tissues in the human brain from radio-frequency measurements obtained by magnetic resonance imaging (MRI).



$\rightsquigarrow$ disease detection
$\rightsquigarrow$ safety standards

ANR Project ELECTRA : IADI (INSERM, Nancy), CHRU Nancy, Healtis (Nancy), ICUBE (Strasbourg), LMR (Reims)

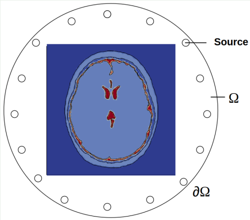
numerical methods for precise reconstruction of electric properties & database with respect to age

Principle :

- strong static magnetic field $\mathbf{B}_0$, e.g. 3T
- excitation of hydrogen protons through radiofrequency (RF) pulse at Larmor frequency, e.g. 128 MHz at 3T $\rightsquigarrow$ field $\mathbf{B}_1 \perp \mathbf{B}_0$
- emission of e.m. signal when protons return to initial state

$\Rightarrow$ picture of biological tissues containing water

Birdcage coil : a typical configuration of RF antenna



$\Omega \rightsquigarrow$ computational domain

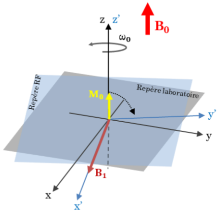
$D \subset \Omega \rightsquigarrow$ measurement area

line source field from 8 or 16 legs (outside D)

MR-EPT : Electrical Properties Tomography with Magnetic Resonance modality

MR sequences for MR-EPT correspond to particular measurements of the rotating RF field $\mathbf{B}_1$ perpendicular to $\mathbf{B}_0$:

- B_1^+ is the rotating component
- B_1^- is the counter-rotating component



What is measured in MR-EPT ?

$$B_1^+ = \frac{B_{1,x} + iB_{1,y}}{2} \text{ in } D$$

Maxwell's equations

time-harmonic Maxwell's equations at fixed angular frequency ω with linear isotropic constitution laws, Ohm's law and scaling wrt electric permittivity ε_0 and magnetic permeability μ_0 in free space

$$\Rightarrow \qquad \mathcal{E}(\mathbf{x},t) = \Re e\left(e^{-i\omega t}\sqrt{\varepsilon_0}\mathbf{E}\right), \; \mathcal{H}(\mathbf{x},t) = \Re e\left(e^{-i\omega t}\sqrt{\mu_0}\mathbf{H}\right)$$

$$\begin{aligned} -ik\varepsilon_r\mathbf{E} - \operatorname{curl}\mathbf{H} &= -\sqrt{\mu_0}\mathbf{J}_s \\ -ik\mu_r\mathbf{H} + \operatorname{curl}\mathbf{E} &= 0 \end{aligned}$$

with source term $\mathbf{J}_s$, wave number $k = \omega\sqrt{\varepsilon_0\mu_0}$ and relative electromagnetic parameters

$$\varepsilon_r = \frac{1}{\varepsilon_0}\left(\varepsilon + \frac{i\sigma}{\omega}\right), \; \mu_r = \frac{\mu}{\mu_0} = 1.$$

Electric field formulation - 2D

Elimination of H :

$$\operatorname{curl} \mu_r^{-1} \operatorname{curl} \mathbf{E} - k^2 \varepsilon_r \mathbf{E} = ik\sqrt{\mu_0}\mathbf{J}_s$$

Two-dimensional transverse magnetic setting and $\mu_r = 1$:

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}, \mathbf{J}_s = \begin{pmatrix} 0 \\ 0 \\ j_s \end{pmatrix} \;\Rightarrow\; -\Delta E - k^2 \varepsilon_r E = \underbrace{ik\sqrt{\mu_0}\, j_s}_{\stackrel{\text{def}}{=} F}$$

Perfect conductor boundary condition in a bounded domain Ω :

$$\begin{cases} -\Delta E - k^2 \varepsilon_r E = F & \text{in } \Omega, \\ E = 0 & \text{on } \partial\Omega. \end{cases}$$

Perturbation due to the presence of an object

Reference configuration : no object $\Rightarrow \varepsilon_r = 1$. Source term F.

$$\left\{ \begin{array}{rcll} -\Delta E^{\mathrm{ref}} - k^2 E^{\mathrm{ref}} & = & F & \text{in } \Omega, \\ E^{\mathrm{ref}} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \quad (1)$$

Configuration with object : assume that $\mathrm{supp}(1-\varepsilon_r) \subset D$ where $D \subset\subset \Omega$. Same source term F.

$$\left\{ \begin{array}{rcll} -\Delta E^{\mathrm{tot}} - k^2 \varepsilon_r E^{\mathrm{tot}} & = & F & \text{in } \Omega, \\ E^{\mathrm{tot}} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \quad (2)$$

Forward problem

Let E^{ref} be the solution to (1) for given F.
For **given** ε_r, find the **scattered field** $E^{\mathrm{sc}} = E^{\mathrm{tot}} - E^{\mathrm{ref}}$, solution to

$$\left\{ \begin{array}{rcll} -\Delta E^{\mathrm{sc}} - k^2 \varepsilon_r E^{\mathrm{sc}} & = & -k^2(1-\varepsilon_r) E^{\mathrm{ref}} & \text{in } \Omega, \\ E^{\mathrm{sc}} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \quad (3)$$

$\hookrightarrow$ well-posedness under appropriate assumptions on k^2 and ε_r

Maxwell's equations $(\mu_r = 1)$ $\rightsquigarrow$ $\operatorname{curl} \mathbf{E} = ik\mathbf{H}$

$$\text{in 2D} : \mathbf{H} = -\frac{i}{k} \mathbf{curl}\, E = -\frac{i}{k} \begin{pmatrix} \partial_y E \\ -\partial_x E \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{C}(E).$$

MR-EPT measurements

$$B_1^+ = \frac{H_x^{\mathrm{sc}} + iH_y^{\mathrm{sc}}}{2} \stackrel{\text{def}}{=} \mathcal{B}(\mathbf{H}^{\mathrm{sc}}) \text{ in } D$$

where $\mathbf{H}^{\mathrm{sc}} = \mathcal{C}(E^{\mathrm{sc}})$ and E^{sc} is the scattered electric field, solution to the forward problem

$$\begin{cases} -\Delta E^{\mathrm{sc}} - k^2 \varepsilon_r E^{\mathrm{sc}} & = & -k^2(1-\varepsilon_r)E^{\mathrm{ref}} & \text{in } \Omega, \\ E^{\mathrm{sc}} & = & 0 & \text{on } \partial\Omega. \end{cases}$$

A first formulation for the inverse problem

Inverse problem as a parameter problem

Let E^{ref} be the solution to (1) for given F.

For **given measurements** f^{data} on $D \subset \Omega$, find ε_r such that the B_1^+-field derived from the scattered field E^{sc}, solution to the forward problem with ε_r, satifies $B^+_{1|D} = f^{\mathrm{data}}$.

Inverse problems are usually ill-posed :

1. lack of existence
2. lack of uniqueness (two different parameters yield the same measurement)
3. lack of continuity (small error in the data $\rightsquigarrow$ large error in the solution)

Uniqueness : an identifiability result

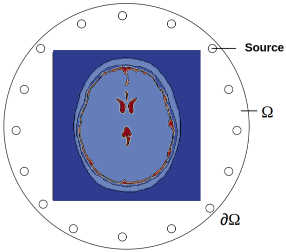
Theorem

Let f_1^{data} and f_2^{data} be two measurements corresponding to the parameters $\varepsilon_{r,1}$ and $\varepsilon_{r,2}$ respectively, and let Assumptions (1–3) be true.

Then, $f_1^{\mathrm{data}} = f_2^{\mathrm{data}}$ on D implies $\varepsilon_{r,1} = \varepsilon_{r,2}$ on Ω.

Assumptions

1. $\mathrm{supp}(F) \cap D = \emptyset$
2. $E^{\mathrm{ref}} \neq 0$ a.e. in D
3. $\mathrm{supp}(1 - \varepsilon_r) \subset\subset D$



Sketch of the proof

Let $E_\ell^{\rm sc} = E_\ell^{\rm sc}(\varepsilon_{r,\ell})$ $(\ell = 1,2)$, and assume $f_1^{\rm data} = f_2^{\rm data}$ in D :

$$H_{x,1}^{\rm sc} + {\rm i} H_{y,1}^{\rm sc} = H_{x,2}^{\rm sc} + {\rm i} H_{y,2}^{\rm sc} \text{ in } D$$

where $\mathbf{H}_\ell^{\rm sc} = -\dfrac{i}{k}\,\mathbf{curl}\, E_\ell^{\rm sc}$ $(\ell = 1,2)$.

Let $v = E_1^{\rm sc} - E_2^{\rm sc}$.

$\Rightarrow \mathrm{Re}(v), \mathrm{Im}(v)$ satisfy the Cauchy-Riemann equations.

$\Rightarrow$ $$\Delta(E_1^{\rm sc} - E_2^{\rm sc}) = 0 \text{ in } D.$$

$E_\ell^{\rm sc}$ is solution of Helmholtz equation with parameter $\varepsilon_{r,\ell}$ $(\ell = 1,2)$:

$$-\Delta E_\ell^{\rm sc} - k^2 \varepsilon_{r,\ell} E_\ell^{\rm sc} = -k^2(1-\varepsilon_{r,\ell})E^{\rm ref}$$

$\Delta(E_1^{\rm sc} - E_2^{\rm sc}) = \Delta v = 0$ in D

$\Rightarrow$ $$\varepsilon_{r,1}E_1^{\rm sc} - \varepsilon_{r,2}E_2^{\rm sc} = -\left(\varepsilon_{r,1} - \varepsilon_{r,2}\right) E^{\rm ref} \text{ in } D.$$

Sketch of the proof - continued

$$\varepsilon_{r,1} E_1^{\mathrm{sc}} - \varepsilon_{r,2} E_2^{\mathrm{sc}} = -\left(\varepsilon_{r,1} - \varepsilon_{r,2}\right) E^{\mathrm{ref}} \text{ in } D$$

Assumption on $\varepsilon_{r,\ell} \Rightarrow \exists B \subset D: \; \varepsilon_{r,\ell} = 1$ in B.

$\Rightarrow$ $$v = E_1^{\mathrm{sc}} - E_2^{\mathrm{sc}} = 0 \text{ in } B \text{ and } \Delta v = 0 \text{ in } D.$$

Unique continuation principle : $v = 0$ in D, i.e. $E_1^{\mathrm{sc}} = E_2^{\mathrm{sc}}$ in D.

$$\left(\varepsilon_{r,1} - \varepsilon_{r,2}\right) \underbrace{E^{\mathrm{tot}}}_{=E^{\mathrm{sc}}+E^{\mathrm{ref}}} = 0 \text{ in } D.$$

It follows from the assumptions on F and E^{ref} that $E^{\mathrm{tot}} \neq 0$ a.e. in D. Thus $\varepsilon_{r,1} = \varepsilon_{r,2}$ a.e. in D.

$\blacksquare$

The inverse problem as contrast source inversion

Numerical resolution of the inverse parameter problem by minimization of a least square functional of the data error $\rightsquigarrow$ evaluation of the functional needs the resolution of the forward problem

$$\begin{cases} -\Delta E^{\mathrm{sc}} - k^2 \varepsilon_r E^{\mathrm{sc}} = -k^2(1-\varepsilon_r)E^{\mathrm{ref}} & \text{in } \Omega, \\ E^{\mathrm{sc}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Difficulty : the operator depends on ε_r

Reformulation of the forward problem

$$\begin{cases} -\Delta E^{\mathrm{sc}} - k^2 E^{\mathrm{sc}} = -k^2(1-\varepsilon_r)\underbrace{(E^{\mathrm{sc}} + E^{\mathrm{ref}})}_{=E^{\mathrm{tot}}} & \text{in } \Omega, \\ E^{\mathrm{sc}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (4)$$

Let $\chi = 1 - \varepsilon_r$ the contrast function and $w = \chi E^{\mathrm{tot}}$ the contrast source.

Linear solution operator $\begin{cases} \mathcal{L}_b : L^2(\Omega) \to H_0^1(\Omega) \\ \qquad w \mapsto E^{\mathrm{sc}} \end{cases}$

where $E^{\mathrm{sc}} \in H_0^1(\Omega)$ is the (variational) solution to (4) with $w = \chi E^{\mathrm{tot}}$.

The inverse problem as contrast source inversion

Let $\mathcal{R}_D : L^2(\Omega) \to L^2(D)$ be the restriction operator to $D \subset \Omega$.

Recall that $\chi = 1 - \varepsilon_r$ and $w = \chi E^{\mathrm{tot}}$.

Operator formulation of the data and state equation :

$$\left| \begin{array}{rcl} B^+_{1|D} = f^{\mathrm{data}} & \Rightarrow & (\mathcal{R}_D \circ \mathcal{B} \circ \mathcal{C} \circ \mathcal{L}_b)(w) = f^{\mathrm{data}} \\ E^{\mathrm{ref}} + E^{\mathrm{sc}} = E^{\mathrm{tot}} & \Rightarrow & \chi\left(E^{\mathrm{ref}} + \mathcal{L}_b(w)\right) = w \end{array} \right.$$

$$\mathcal{D}^{\mathrm{data}} \stackrel{\mathrm{def}}{=} \mathcal{R}_D \circ \mathcal{B} \circ \mathcal{C} \circ \mathcal{L}_b$$

Inverse problem as contrast source inversion (CSI)

Let E^{ref} be the solution to (1) for given F.

For **given measurements** f^{data} on $D \subset \Omega$, find (χ, w) defined on Ω, such that

$$\left\{ \begin{array}{rcll} \mathcal{D}^{\mathrm{data}}(w) & = & f^{\mathrm{data}} & \textbf{[data equation]} \\ \chi\left(E^{\mathrm{ref}} + \mathcal{L}_b(w)\right) & = & w & \textbf{[state equation]} \end{array} \right.$$

Resolution of the inverse problem by minimization

Cost function

$$\mathcal{F}(\chi, w) = \mathcal{F}^{\mathrm{data}}(w) + \mathcal{F}^{\mathrm{state}}(\chi, w) \longrightarrow \min$$

$$\text{with} \quad \begin{aligned} \mathcal{F}^{\mathrm{data}}(w) &= \frac{\eta^{\mathrm{data}}}{2} \| f^{\mathrm{data}} - \mathcal{D}^{\mathrm{data}}(w) \|^2_{0,D}, \\ \mathcal{F}^{\mathrm{state}}(\chi, w) &= \frac{\eta^{\mathrm{state}}}{2} \| \chi (E^{\mathrm{ref}} + \mathcal{L}_b(w)) - w \|^2_{0,D}, \end{aligned}$$

for any $\chi \in L^\infty(\Omega)$ and $w \in L^2(\Omega)$ and normalization constants $\eta^{\mathrm{data}} > 0$ and $\eta^{\mathrm{state}} > 0$.

Two-step iterative method :

- update w_n by a conjugate gradient (Polak-Ribière) method $\rightsquigarrow$ w_{n+1}
- compute $E^{\mathrm{tot}}_{n+1} = E^{\mathrm{ref}} + \mathcal{L}_b(w_{n+1})$
- update χ_n from knowledge of E^{tot}_{n+1} in order to satisfy the state equation

Numerical results

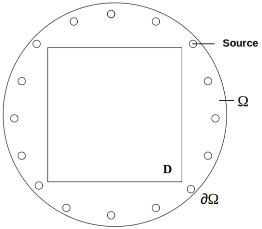


Figure – Configuration of a birdcage coil

Line source field

$$j_s(\mathbf{x}) = \sqrt{I} \sum_{\ell=1}^{16} \frac{e^{ikr_\ell}}{\sqrt{r_\ell}}$$

with intensity I and $r_\ell = \|\mathbf{x} - \mathbf{c}_\ell\|$ the distance to the center $\mathbf{c}_\ell$ of the ℓth leg.

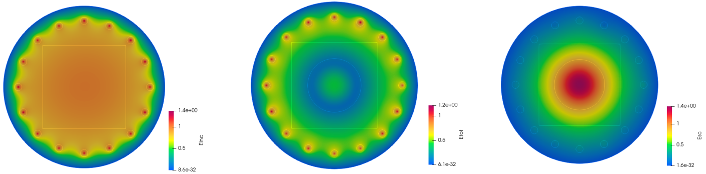


Figure – E^{ref} (without object, right), E^{tot} (with object, middle), E^{sc} (scattered field, right), P1 FEM - FreeFem++

Numerical results : academic configuration

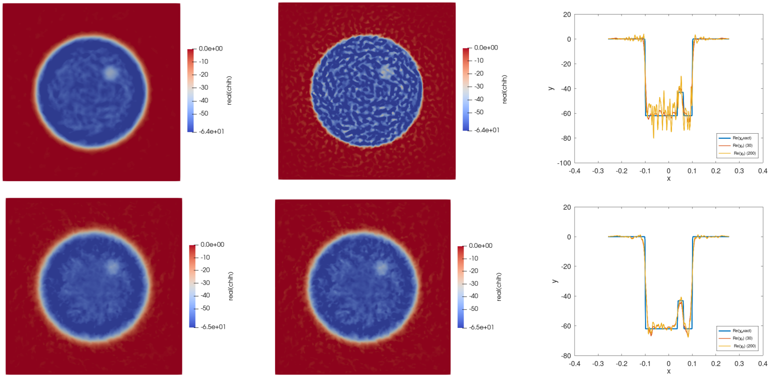


Figure – Reconstruction at iteration 30 (left) and 200 (middle). View 1d at line $y = x$ (right). Real part of ε_r. Upper line : noiseless data without regularization. Lower line : 5% noisy data with Tikhonov regularization.

Realistic 2D configuration of the humain brain

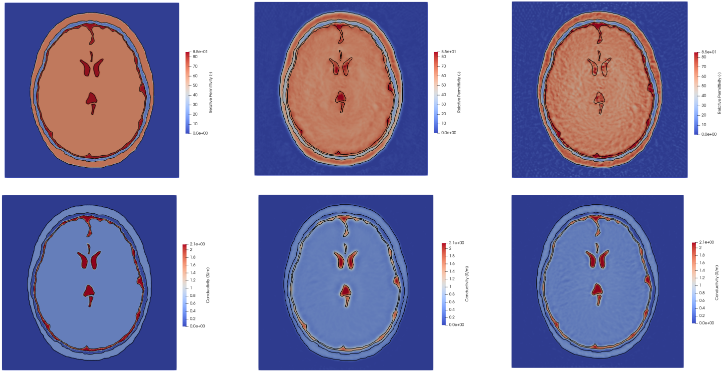


Figure – Ground truth (left). Reconstruction from noiseless data at iteration 50 (middle) and 100 (right). Upper line : permittivity. Lower line : conductivity.

Realistic 2D configuration of the humain brain

Data mesh : ca. 100 000 nodes (200 000 elements)
CSI mesh : ca. 60 000 nodes (120 000 elements)

Number of measurements in D **:** ca. 34 000.
Computational time (PC) : ca. 2h.

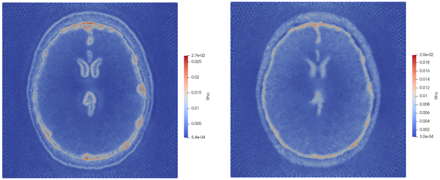
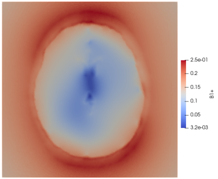


Figure – Data error ρ at iteration 50 (left) and 100 (right).

L^2**-error on permittivity :** 13% at iteration 100.
L^2**-error on conductivity :** 27% at iteration 100.

Conclusion and on-going work

- Identifiability of EPs ε and σ from realistic B_1^+-data.
- Reconstruction of ε and σ in various academic and realistic configurations.



$|B_1^+|$

Future work

- (Multiplicative) regularization of CSI [cf. Balidemaj et al 2016]
- Phaseless data : $|B_1^+|$ instead of full B_1^+ data [cf. Arduino et al. 2018]
- Experimental data of a phantom [IADI, Nancy]

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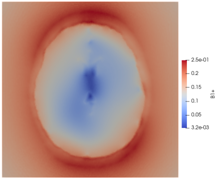

$|B_1^+|$

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Thank you and Happy Birthday, Patrick !