The $\boldsymbol{A} - \varphi$ harmonic formulation in eddy current problems: T-coercivity, perturbed formulation and error estimations

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Outline

The eddy current problem

Find the electric field $\boldsymbol{E}$, the magnetic flux $\boldsymbol{B}$, the magnetic field $\boldsymbol{H}$ and the eddy current $\boldsymbol{J}_e$ such that

$$\begin{cases} \operatorname{curl} \boldsymbol{E} &= -j\omega \boldsymbol{B}, \text{ in } \Omega \subset \mathbb{R}^3, \\ \operatorname{curl} \boldsymbol{H} &= \boldsymbol{J}_s + \boldsymbol{J}_e, \text{ in } \Omega \\ \operatorname{div} \boldsymbol{B} &= 0, \text{ in } \Omega \end{cases} \tag{1}$$

where Ω is bd simply connected with a connected bdy, $\boldsymbol{J}_s$ is the divergence free current density, $j^2 = -1$ and ω the pulsation, with the constitutive laws

$$\boldsymbol{B} = \mu \boldsymbol{H} \text{ in } \Omega \text{ and } \boldsymbol{J}_e = \sigma \boldsymbol{E} \text{ in } \Omega_c \subset\subset \Omega, \tag{2}$$

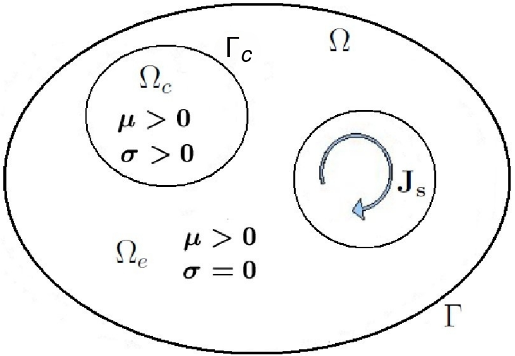
where μ denotes the magnetic permeability and σ the electric conductivity. The boundary conditions are respectively

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma = \partial\Omega, \tag{3}$$

$$\boldsymbol{J}_e \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_c = \partial\Omega_c, \tag{4}$$

where $\boldsymbol{n}$ stands for the unit outward normal to Ω or Ω_c.

An illustration of the geometry



Goals

- Recall the standard variational formulation,
- Give its saddle point formulation,
- Deduce a perturbed formulation (use the explicit form of the coercivity operator T),
- Perform an a posteriori error analysis of the discrete perturbed formulation.

The harmonic $\boldsymbol{A}-\varphi$ formulation

Based on the introduction of a vector potential $\boldsymbol{A}$ in Ω and a scalar potential φ in Ω_c such that:

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \text{ in } \Omega \text{ and } \boldsymbol{E} = -j\omega \boldsymbol{A} - \nabla\varphi \text{ in } \Omega_c.$$

(1), (3), (4) $\Rightarrow$ the harmonic $\boldsymbol{A}-\varphi$ formulation:

$$\boxed{\begin{array}{l} \operatorname{curl}\left(\mu^{-1}\operatorname{curl}\boldsymbol{A}\right) + \sigma\Big(j\,\omega\,\boldsymbol{A} + \nabla\varphi\Big) = \boldsymbol{J}_s \text{ in } \Omega, \\ \operatorname{div}(\sigma(j\,\omega\,\boldsymbol{A} + \nabla\varphi)) = 0 \text{ in } \Omega_c, \\ \boldsymbol{A}\times\boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \sigma(j\omega\boldsymbol{A} + \nabla\varphi)\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_c. \end{array}}$$

The Coulomb gauge: $\operatorname{div}\boldsymbol{A} = 0$ & $\int_{\Omega_c}\varphi = 0 \Rightarrow$ uniqueness of these potentials.

Weak formulation

Set

$$\mathbf{K}_N(\mathcal{D}) = \left\{ \boldsymbol{F} \in \mathbf{H}_0(\mathrm{curl}, \mathcal{D}) : \mathrm{div}\, \boldsymbol{F} = 0 \right\},$$
$$H^1_{zmv}(\mathcal{D}) = \left\{ f \in H^1(\mathcal{D}) : (f, 1)_{\mathcal{D}} = 0 \right\},$$

$$a((\boldsymbol{A}, \varphi), (\boldsymbol{A}', \varphi')) = \left(\mu^{-1} \,\mathrm{curl}\, \boldsymbol{A}, \mathrm{curl}\, \boldsymbol{A}'\right)_\Omega + j\omega^{-1} \left(\sigma(j\omega \boldsymbol{A} + \nabla\varphi), (j\omega \boldsymbol{A}' + \nabla\varphi')\right)_{\Omega_c}.$$

Find $(\boldsymbol{A}, \varphi) \in \mathbf{K}_N(\Omega) \times H^1_{zmv}(\Omega_c)$ such that

$$\boxed{a((\boldsymbol{A}, \varphi), (\boldsymbol{A}', \varphi')) = (\boldsymbol{J}_s, \boldsymbol{A}')_\Omega, \quad \forall (\boldsymbol{A}', \varphi') \in \mathbf{K}_N(\Omega) \times H^1_{zmv}(\Omega_c).} \tag{6}$$

Theorem 2.1 of [Creusé et al:12] ensures the existence and uniqueness of the weak solution $(\boldsymbol{A}, \varphi)$ of this problem.

E. Creusé, S. Nicaise, Z. Tang, Y. Le Menach, N. Nemitz, and F. Piriou.
Math. Models Methods Appl. Sci., 22(5):1150028, 30, 2012.

The saddle point formulation

The Coulomb gauge $\operatorname{div} \boldsymbol{A} = 0$ leads to some numerical difficulties, hence we transform the previous FV into a saddle point pb.
For $\gamma > 0$ arbitrary, introduce the bd sesquilinear form on $\mathbb{V} := \mathbf{H}_0(\mathrm{curl};\Omega) \times H^1_{zmv}(\Omega_c) \times H^1_0(\Omega)$ by

$$\mathcal{A}_\gamma((\boldsymbol{v},\varphi,q),(\boldsymbol{v}',\varphi',q')) = a((\boldsymbol{v},\varphi),(\boldsymbol{v}',\varphi')) + \gamma(\overline{(\boldsymbol{v}',\nabla q)_\Omega} + (\boldsymbol{v},\nabla q')_\Omega).$$

The saddle point pb: Find $(\boldsymbol{A},\varphi,p_A) \in \mathbb{V}$ such that

$$\boxed{\mathcal{A}_\gamma((\boldsymbol{A},\varphi,p_A),(\boldsymbol{v}',\varphi',q')) = (\boldsymbol{J}_s,\boldsymbol{v}')_\Omega, \quad \forall(\boldsymbol{v}',\varphi',q') \in \mathbb{V}.} \tag{7}$$

Lemma

$(\boldsymbol{A},\varphi)$ solves (6) if and only if $(\boldsymbol{A},\varphi,0)$ solves (7).

The saddle point formulation: well-posedness via the T-coercivity

For $(\boldsymbol{v}, \psi, q) \in \mathbb{V}$, we split $\boldsymbol{v} = \boldsymbol{k} + \nabla\phi$, where $(\boldsymbol{k}, \phi) \in \mathbf{K}_N(\Omega) \times H_0^1(\Omega)$, and define the operator

$$T \;:\; (\boldsymbol{k} + \nabla\phi, \psi, q) \mapsto (\boldsymbol{k} + \nabla q, \psi^*, \phi),$$

with $j\omega\boldsymbol{v} + \nabla\psi = j\omega(\boldsymbol{k} + \nabla q) + \nabla\psi^*$ so that

$$\begin{aligned}\mathcal{A}_\gamma((\boldsymbol{v}, \psi, q), T(\boldsymbol{v}, \psi, q)) &= (\mu^{-1}\operatorname{curl}\boldsymbol{k}, \operatorname{curl}\boldsymbol{k})_\Omega + \gamma\|\nabla\phi\|_\Omega^2 \\ &+\gamma\|\nabla q\|_\Omega^2 + j\omega^{-1}\|\sqrt{\sigma}(j\omega\boldsymbol{v} + \nabla\psi)\|_{\Omega_c}^2.\end{aligned}$$

This implies that $\mathcal{A}_\gamma((\boldsymbol{v}, \psi, q), T(\boldsymbol{v}, \psi, q))$ is coercive on $\mathbb{V}$.
Further $T^2 = \mathbb{I}$ so T is an iso from $\mathbb{V}$ into itself, hence (7) is well-posed.

P. Ciarlet Jr.
T-coercivity: A practical tool for the study of variational formulations in Hilbert spaces, book in preparation.

Equivalent formulation using the mapping T

Set $F(\boldsymbol{v}', \varphi', q') = (\boldsymbol{J}_s, \boldsymbol{v}')_\Omega$, then as T is an iso,
$\mathcal{A}_\gamma((\boldsymbol{A}, \varphi, p_A), (\boldsymbol{v}', \varphi', q')) = F(\boldsymbol{v}', \varphi', q'), \forall (\boldsymbol{v}', \varphi', q') \in \mathbb{V} \iff$
$\mathcal{A}_\gamma((\boldsymbol{A}, \varphi, p_A), T(\boldsymbol{v}', \varphi', q')) = F(T(\boldsymbol{v}', \varphi', q')), \forall (\boldsymbol{v}', \varphi', q') \in \mathbb{V}.$

But $\operatorname{div} \boldsymbol{J}_s = 0 \Rightarrow F(T(\boldsymbol{v}', \varphi', q')) = (\boldsymbol{J}_s, \boldsymbol{v}')_\Omega, \forall (\boldsymbol{v}', \varphi', q') \in \mathbb{V}.$
This yields the equivalent formulation of (7):

$$\boxed{a((\boldsymbol{A}, \varphi), (\boldsymbol{v}', \varphi')) + \gamma(\nabla \phi_A, \nabla \phi')_\Omega = (\boldsymbol{J}_s, \boldsymbol{v}')_\Omega,}$$

for all $(\boldsymbol{v}', \varphi') \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times H^1_{zmv}(\Omega_c)$,
when $\boldsymbol{v}' = \boldsymbol{k}' + \nabla \phi'$ and $\boldsymbol{A} = \boldsymbol{k}_A + \nabla \phi_A$.
Note that the solution is independent of γ, but the red term is problematic for numerical purposes. So we replace it by

$$\gamma(\boldsymbol{A}, \boldsymbol{v}')_\Omega,$$

which now yields a solution that depends on γ.

Perturbed formulation

Given an arbitrary positive real number γ, the perturbed formulation of (6) consists in looking for $(\boldsymbol{A}_\gamma, \varphi_\gamma) \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times H^1_{zmv}(\Omega_c)$ solution of

$$\boxed{a_\gamma((\boldsymbol{A}_\gamma, \varphi_\gamma), (\boldsymbol{A}', \varphi')) = (\boldsymbol{J}_s, \boldsymbol{A}')_\Omega, \quad \forall (\boldsymbol{A}', \varphi') \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times H^1(\Omega_c),} \tag{8}$$

where

$$a_\gamma((\boldsymbol{A}, \varphi), (\boldsymbol{A}', \varphi')) = a((\boldsymbol{A}, \varphi), (\boldsymbol{A}', \varphi')) + \gamma(\boldsymbol{A}, \boldsymbol{A}')_\Omega.$$

As $\mathrm{div}\, \boldsymbol{J}_s = 0$, by taking $A' = \nabla\psi$, with $\psi \in H^1_0(\Omega)$ and $\varphi' = -(j\omega)^{-1}\psi$ in (8), we obtain

$$\mathrm{div}\, \boldsymbol{A}_\gamma = 0.$$

Note that

$$\|\boldsymbol{A}_\gamma - \boldsymbol{A}\|_{H_0(\mathrm{curl},\Omega)} + \|\varphi_\gamma - \varphi\|_{1,\Omega} \lesssim \gamma \|\boldsymbol{J}_s\|_\Omega.$$

Since γ is assumed to be small, without loss of generality we can assume that $\gamma \leq 1$.

Discrete perturbed formulation

Assume given a finite dimensional space $\mathbf{X}_h \subset \mathbf{H}_0(\mathrm{curl}, \Omega)$ and a finite dimensional space $V_h \subset H_0^1(\Omega)$ such that

$$\nabla V_h \subset \mathbf{X}_h.$$

We introduce the spaces

$$V_{c,h} = \{\varphi \in H^1(\Omega_c) : \exists \psi_h \in V_h \text{ such that } \varphi = \psi_h \text{ on } \Omega_c\},$$
$$\tilde{V}_{c,h} = \{\varphi \in V_{c,h} : \int_{\Omega_c} \varphi(x)\,dx = 0\}.$$

The discrete approx. of (8): Find $(\boldsymbol{A}_{\gamma,h}, \varphi_{\gamma,h}) \in \mathbf{X}_h \times \tilde{V}_{c,h}$ s. t.

$$\boxed{a_\gamma((\boldsymbol{A}_{\gamma,h}, \varphi_{\gamma,h}), (\boldsymbol{A}'_h, \varphi'_h)) = (\boldsymbol{J}_s, \boldsymbol{A}'_h)_\Omega, \quad \forall (\boldsymbol{A}'_h, \varphi'_h) \in \mathbf{X}_h \times V_{c,h}.} \qquad (9)$$

The error estimator

The polyhedron $\overline{\Omega}$ is now assumed to be triangulated by a shape regular family of meshes $(\mathcal{T}_h)_h$, made of (closed) simplices T that is conform with respect to Ω_c. Then we introduce the error estimator

$$\eta^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2, \quad \eta_T^2 = \sum_{\ell=1}^{4} \eta_{T;\ell}^2 + \sum_{F\subset\partial T} \sum_{\ell=1}^{3} \eta_{F;\ell}^2, \forall T \in \mathcal{T}_h,$$

$$\begin{aligned}
\eta_{T;1} &= h_T \|\pi_h \boldsymbol{J}_s - \operatorname{curl}(\mu^{-1} \operatorname{curl} \boldsymbol{A}_{\gamma,h}) - \gamma \boldsymbol{A}_{\gamma,h} - \sigma(j\omega \boldsymbol{A}_{\gamma,h} + \nabla\varphi_{\gamma,h})\|_T,\\
\eta_{T;2} &= h_T \|\boldsymbol{J}_s - \pi_h \boldsymbol{J}_s\|_T, \quad \eta_{T;3} = h_T \|\operatorname{div}(\sigma(j\omega \boldsymbol{A}_{\gamma,h} + \nabla\varphi_{\gamma,h}))\|_T,\\
\eta_{F;1} &= h_F^{\frac{1}{2}} \| \left[\mu^{-1} \operatorname{curl} \boldsymbol{A}_{\gamma,h} \times \boldsymbol{n}_F\right]_F \|_F,\\
\eta_{F;2} &= h_F^{\frac{1}{2}} \| \left[(\sigma(j\omega \boldsymbol{A}_{\gamma,h} + \nabla\varphi_{\gamma,h}) \cdot \boldsymbol{n}_F\right]_F \|_F,\\
\eta_{T;4} &= \sqrt{\gamma} h_T \|\operatorname{div} \boldsymbol{A}_{\gamma,h}\|_T, \quad \eta_{F;3} = \sqrt{\gamma} h_F^{\frac{1}{2}} \| \left[\boldsymbol{A}_{\gamma,h} \cdot \boldsymbol{n}_F\right]_F \|_F,
\end{aligned}$$

where $\boldsymbol{n}_F =$ one fixed unit normal vector to F and $[\cdot]_F =$ jump of the quantity through F.

Reliability

For the reliability, we require the following (weak) assumption

$$\mathcal{ND}_0(\mathcal{T}_h) \cap \mathbf{H}_0(\mathrm{curl}, \Omega) \subset \mathbf{X}_h \text{ and } PP^1(\mathcal{T}_h) \subset V_h,$$

where $\mathcal{ND}_0(\mathcal{T}_h)$ are the Nédélec FE space of lowest degree on $\mathcal{T}_h$ and for any $\ell \in \mathbb{N}$,

$$PP^\ell(\mathcal{T}_h) = \{u \in L^2(\Omega) : u_{|T} \in \mathbb{P}^\ell(T), \forall T \in \mathcal{T}_h\}.$$

Under these assumptions, for any $\varphi \in L^2(\Omega)$ and $\boldsymbol{A}' \in L^2(\Omega)$, their Clément type interpolant $I_{\mathrm{Cl}}\varphi$ and $P_{\mathrm{Cl}}\boldsymbol{A}'$ are well-defined and interpolation error estimates hold. These estimates combined with the Galerkin orthogonality yield

Theorem

Let $\boldsymbol{e_A} = \boldsymbol{A}_\gamma - \boldsymbol{A}_{\gamma,h}$, $e_\varphi = \varphi_\gamma - \varphi_{\gamma,h}$. Then

$$\|\mu^{-1/2} \,\mathrm{curl}\, \boldsymbol{e_A}\|_\Omega + \sqrt{\gamma}\|\boldsymbol{e_A}\|_\Omega + \|\sqrt{\frac{\sigma}{\omega}}(j\omega \boldsymbol{e_A} + \nabla e_\varphi)\|_{\Omega_c} \lesssim (1 + \omega^{-\frac{1}{2}})\eta.$$

Efficiency

For the efficiency, we require the following assumptions that there exists $\ell \in \mathbb{N}^*$ such that

$$\mathbf{X}_h \subset PP^\ell(\mathcal{T}_h)^3 \text{ and } V_h \subset PP^\ell(\mathcal{T}_h),$$

and

$$\sigma, \mu \in PP^\ell(\mathcal{T}_h).$$

As usual, the local efficiency is based on the use of bubble fcts, inverse inequalities and integration by parts.

Theorem

For all $T \in \mathcal{T}_h$, we have

$$\eta_T \lesssim \sqrt{\gamma}\|\boldsymbol{e_A}\|_{\omega_T} + \omega^{\frac{1}{2}}\|\sqrt{\frac{\sigma}{\omega}}(j\omega\boldsymbol{e_A} + \nabla e_\varphi)\|_{\omega_T} + \|\mu^{-1}\operatorname{curl}\boldsymbol{e_A}\|_{\omega_T} + \sum_{T' \subset \omega_T} \eta_{T';2}. \tag{10}$$


Happy Birthday